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On the Generalized Gergonne Point and Beyond

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Abstract. In this paper we further extend the generalization of the concept of Gergonne point for circles concentric to the inscribed circle. Given a triangle $V_1V_2V_3$, a point I and three arbitrary directions q_1, q_2, q_3 from I, we find a distance $x = IQ_1 = IQ_2 = IQ_3$ along these directions, for which the three cevians V_iQ_i are concurrent. Types and number of solutions, which can be obtained by the common intersection points of three conics, are also discussed in detail.

1. Introduction

The Gergonne point is a well-known center of the triangle. It is the intersection of the three cevians defined by the touch points of the inscribed circle [3]. Konečný [1] has generalized this to circles concentric with the inscribed circle. Let $\mathcal{C}(I)$ be a circle with center I, the incenter of triangle $V_1V_2V_3$. Let Q_1, Q_2, Q_3 be the points of intersection of $\mathcal{C}(I)$ with the lines from I that are perpendicular to the sides V_2V_3, V_3V_1, V_1V_2 respectively. Then the lines V_iQ_i, $i = 1, 2, 3$, are concurrent (see Figure 1).

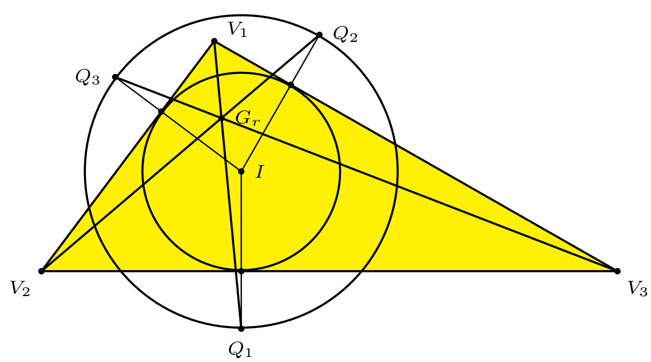

Figure 1. Lines V_iQ_i are also concurrent for circles concentric to the inscribed circle

The first question naturally arises: if the radius of the circle is altered, what will be the locus of the point G_r? Boyd and Raychowdhury [4] computed the convex coordinates of G_r, from which it is clear that the locus is a hyperbola.

Now instead of the inscribed circle consider an inscribed conic (see Figure 2). The lines V_iQ_i, $i = 1, 2, 3$, are still concurrent, at a point called the Brianchon point of the conic (c.f. [5]). There are infinitely many inscribed conics, thus the center

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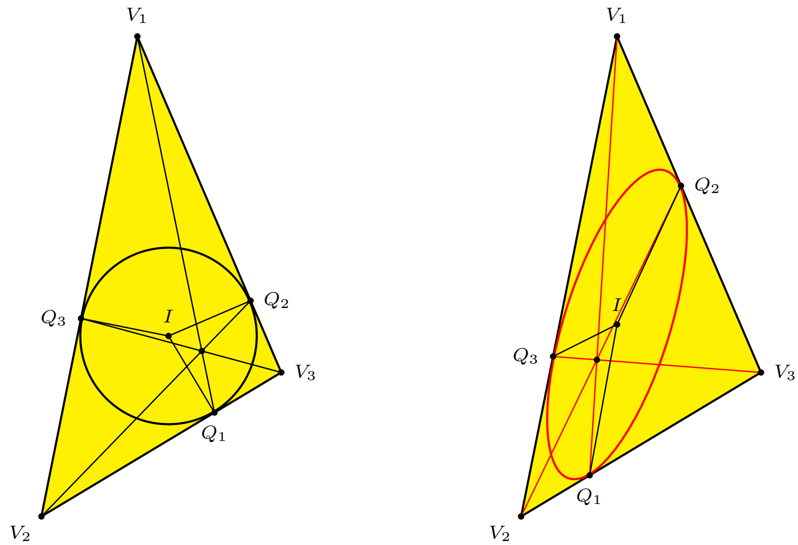

Figure 2. Inscribed ellipse and its directions IQ_i

I and directions q_i (corresponding to the line connecting I and Q_i) can be chosen in many ways, but not arbitrarily. Note that the center completely determines the inscribed conic and the points of tangency Q_1, Q_2, Q_3.

Using these directions we generalize the concept of concentric circles: given a triangle $V_1V_2V_3$ and an inscribed conic with center I and touch points Q_1, Q_2, Q_3, consider the three lines q_i connecting I and Q_i respectively. A circle with center I has to be found which meets the lines q_i at $\bar{Q}_i$ such that the lines $V_i\bar{Q}_i$, $i = 1, 2, 3$, are concurrent. In fact, as we will see in the next section, we do not have to restrict ourselves in terms of the position of the center and the given directions.

2. The general problem and its solution

The general problem can be formulated as follows: given a triangle $V_1V_2V_3$, a point I and three arbitrary directions q_i, find a distance $x = IQ_1 = IQ_2 = IQ_3$ along these directions, for which the three cevians V_iQ_i are concurrent. In general these lines will not meet in one point (see Figure 3): instead of one single center G we have three different intersection points G_{12}, G_{13} and G_{23}.

In the following theorem we will prove that altering the value x, the points G_{12},G_{13} and G_{23} will separately move on three conics. If there is a solution to our generalized problem, it would mean that these conics have to meet in one common point. It is easy to observe that each pair of conics have two common points at I and a vertex of the triangle. Here we prove that the other two intersection points can be common for all the three conics. Previously mentioned special cases are excluded from this point.

Theorem 1. *Let V_1, V_2, V_3 and I be four points in the plane in general positions. Let q_1, q_2, q_3 be three different oriented lines through I ($V_i \notin q_i$). There exist*

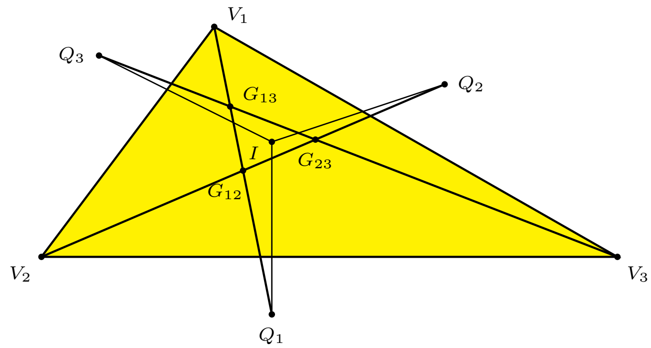

Figure 3. For arbitrary directions and distance, cevians V_iQ_i are generally not concurrent, but meet at three different points

at most two values $x \in \mathbb{R}\backslash\{0\}$ *such that for points* Q_i *along the lines* q_i *with* $IQ_1 = IQ_2 = IQ_3 = x$*, the lines* V_iQ_i *are concurrent.*

Proof. For a real number x and $i = 1, 2, 3$, let $Q_i(x)$ be a point on q_i for which $IQ_i(x) = x$. The correspondences $Q_i(x) \leftrightarrow Q_j(x)$ define perspectivities $(q_i) \overline{\overline{\wedge}} (q_j)$, $(i \neq j)$.

Now let $l_i(x)$ be the line connecting V_i and $Q_i(x)$. The correspondences $l_i(x) \leftrightarrow l_j(x)$ define projectivities $(V_i) \overline{\wedge} (V_j)$, $(i \neq j)$. The intersection points of corresponding lines of these projectivities lie on three conics:

$$\begin{aligned} (V_1) \overline{\wedge} (V_2) &\Rightarrow c_3 \\ (V_1) \overline{\wedge} (V_3) &\Rightarrow c_2 \\ (V_2) \overline{\wedge} (V_3) &\Rightarrow c_1. \end{aligned}$$

We find the intersection points of these conics. Since $Q_i(0) = I$, then $I \in c_i, (i = 1, 2, 3)$. $V_3 \in c_1 \cap c_2$, $V_2 \in c_1 \cap c_3$ and $V_1 \in c_2 \cap c_3$ also hold. Denote the other two intersection points of c_1 and c_2 by S_1 and S_2, *i.e.*,

$$c_2 \cap c_3 = \{I, V_1, S_1, S_2\}.$$

The points S_1 and S_2 can be real and distinct, real and identical, or imaginary in pair.

(i) If they are real and distinct, then for some x_1 and x_2,

$$\begin{aligned} S_1 = l_1(x_1) \cap l_2(x_1) = l_1(x_1) \cap l_3(x_1) \quad &\Rightarrow \quad S_1 = l_2(x_1) \cap l_3(x_1) \\ S_2 = l_1(x_2) \cap l_2(x_2) = l_1(x_2) \cap l_3(x_2) \quad &\Rightarrow \quad S_2 = l_2(x_2) \cap l_3(x_2) \end{aligned}$$

which immediately yields $S_1, S_2 \in c_1$ as well.

(ii) If they are identical, then for the unique x,

$$S = l_1(x) \cap l_2(x) = l_1(x) \cap l_3(x) \quad \Rightarrow \quad S = l_2(x) \cap l_3(x)$$

which yields $S \in c_1$.

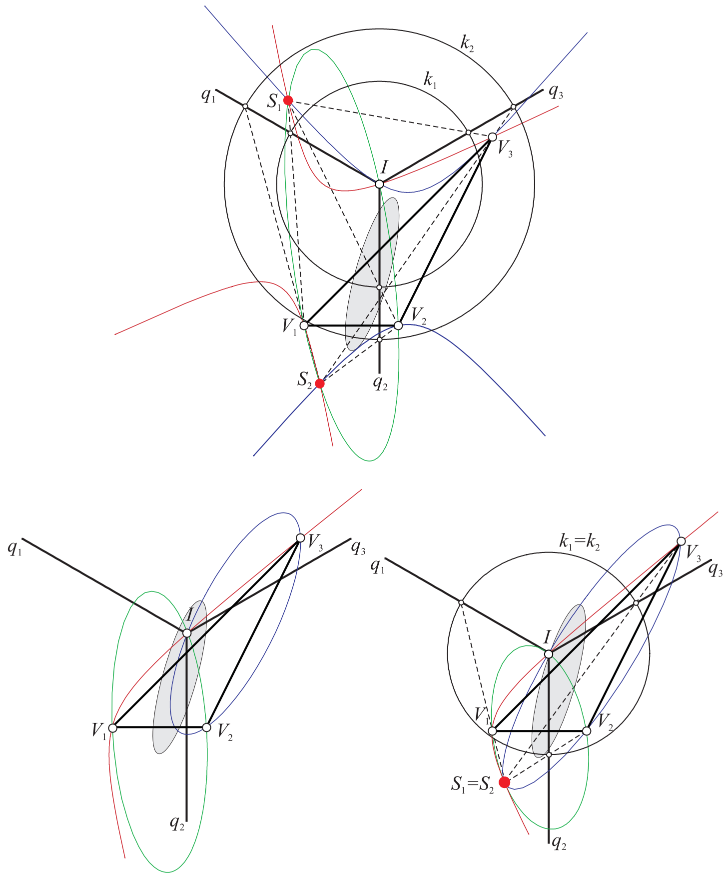

Figure 4. Given a triangle $V_1V_2V_2$ and directions $q_i(i = 1, 2, 3)$ there can be two different real solutions (upper figure), two coinciding solutions (bottom right) and two imaginary solutions (bottom left). Cevians are plotted by dashed lines. The type of solutions depends on the relative position of I to the shaded conic. The three conic paths of G_{12} (green), G_{13} (red) and G_{23} (blue) are also shown. (This figure is computed and plotted by the software *Mathematica*)

(iii) If the points S_1 and S_2 are the pair of imaginary points there are no real number x for which the lines V_iQ_i are concurrent. □

Figure 4 shows the three different possibilities mentioned in the proof. If the triangle and the directions q_i, $i = 1, 2, 3$, are fixed, then the radius of the circle can be obtained by the solutions of a quadratic equation in which the only unknown is the point I. The type of the solutions depends on the discriminant, which is a quadratic function of I. This means that for every triangle and triple of directions there exists a conic which separates the possible positions of I in the following way: if I is outside the conic (discriminant > 0) then there are two different real solutions, if I is on the conic (discriminant $= 0$) then there are two coinciding real solutions, while if I is inside the conic (discriminant < 0) then there are two imaginary solutions. This conic is also shown in Figure 4.

Remarks. (1) Note that there are no further restrictions for the positions of the center and the directed lines. The center can even be outside the reference triangle.

(2) According to the projective principles in the proof, the statement remains valid if we replace the condition $IQ_1 = IQ_2 = IQ_3 = x$ with the more condition that the ratios of these lengths be fixed.

3. Further research

The conics c_i, $i = 1, 2, 3$, play a central role in the proof. The affine types of these conics however, can only be determined by analytical approach or by closer study the type of involutive pencils determined by cevians. It is also a topic of further research how the types of solutions depend on the ratios mentioned in Remark 2. The exact representation of the length of the radius by the given data can also be discussed analytically in a further study.

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