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Singular Points on the Pedal Surfaces of (1,2) Congruences - Type I

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Abstract

In this article the pedal surfaces of (1,2) congruences - type I are classified according to the number and kind of singular points.

Sažetak

U ovom su radu prema broju i vrsti singularnih točaka klasificirane nožišne plohe (1,2) kongruencija tipa I.

1. Introduction

In this paper I would like to focus on the parts of [2] where the program *Mathematica* was used for symbolic computations and plotting. The *Mathematica* notebook `pedals.nb` with functions for computing and plotting is enclosed.

In [2] the full classification of the pedal surfaces of the (1,2) congruences is given in the following way: The congruences of 1st order and 2nd class are classified according to the section with the plane at infinity. Eight types of the congruences have been obtained. The basic classification of the pedal surfaces is obtained according to the type of the primary congruence. It is shown that there are no real simple lines on the first type surfaces, whereas on the surfaces of the II, III, IV and V type there are 2, 3, 4 and 5 real simple lines, respectively. For the rest of the types it is shown that the pedal surface breaks up into the general cubic (types VI and VII), or ruled cubic (type VIII), and the plane. The full classification of the proper quartics of the I-V type surfaces is made according to the number and kind of their singular points. Eighty subtypes are obtained ($\mathrm{I}_{(1-16)}$, $\mathrm{II}_{(1-17)}$, $\mathrm{III}_{(1-8)}$, $\mathrm{IV}_{(1-28)}$, $\mathrm{V}_{(1-11)}$).

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2. The pedal surfaces of (1,2) congruences - type I

2.1 (1,2) congruences - type I

In three-dimensional projective space one (n, m) congruence is a double infinite line system. n is an *order* and m is a *class* of the congruence. Order is the number of congruence lines through a given point, and class is the number of congruence lines in a given plane.

According to [8, p. 37] (1,2) congruence of lines may be regarded as the system of lines meeting two directing curves, a conic c and a straight line d with only one common point O.

According to [2, p. 6-9] the (1,2) congruence of the type I contain a pair of imaginary lines at infinity. They are directed by an ellipse c and a finite line d. In the homogeneous Cartesian coordinate system the directed curves of these congruences (Fig. 1) can be given by the following equations:

$$d \ldots \begin{cases} x = 0 \\ y = 0 \end{cases}$$

$$c \ldots \begin{cases} \gamma \ldots & Ax + By + z = 0, & A, B \in R \\ \chi \ldots & x^2 + ay^2 + bxw + cyw = 0, & a, b, c \in R,\ a > 0,\ ab^2 + c^2 \neq 0. \end{cases} \tag{1}$$

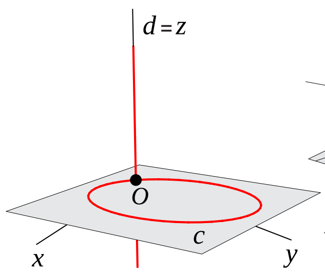


Figure 1

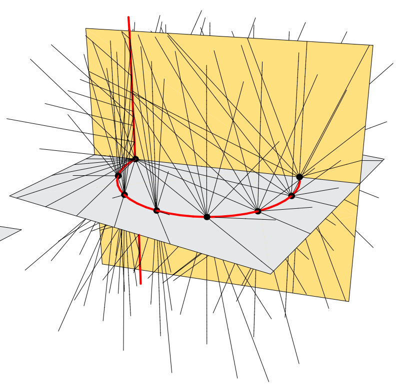


Figure 2

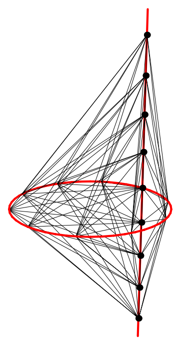


Figure 3

The rays of the congruence can be constructed as:
- the pencils of lines in the planes of the pencil of planes $[d]$ (Fig. 2), or
- the generators of the cones with the vertices on the line d (Fig. 3).

2.2 The pedal surfaces of (1,2) congruences - type I

The locus of the feet of perpendiculars drawn from any fixed finite point P, called the *pole*, on the rays of (n, m) congruence is called the *pedal surface* of that congruence.

The pedal surface of (1,2) congruence for the pole P is the image of the plane at infinity given by quartic inversion [3] with regard to this congruence and any sphere with the center P.

It is a quartic (4th order surface) which contains a double straight line d, a conic c and the absolute conic. If the surface belongs to the type I there are no real simple lines on it and there are at most three double points not lying on the double line.

There are two cases when this pedal surface degenerates:

1. If c is a circle in the plane perpendicular to d and $P \in d$, then the pedal surface breaks up into the sphere and the pair of isotropic planes through d.

2. If c is a circle in the plane perpendicular to the line d and OP is the diameter of c, then the pedal surface is a ruled quartic with double lines c, d and imaginary generators. [2, p. 13-22].

2.3 Construction

The basic theorem for the constructive treatment of the pedal surfaces - type I is:

Theorem 1 *If the plane $\delta_C \in [d]$ cuts the conic c at the point C and P' is the foot of P in the plane $\delta_C \in [d]$, then the plane $\delta_C \in [d]$ cuts the pedal surface along the double straight line d and the circle f, where*

- *if $C \neq P'$, the circle f has the diameter $\overline{CP'}$,*

- *if $C = P'$, the circle f breaks up into the pair of isotropic lines through C.*

Proof. [2, p. 20] (see Fig. 4)

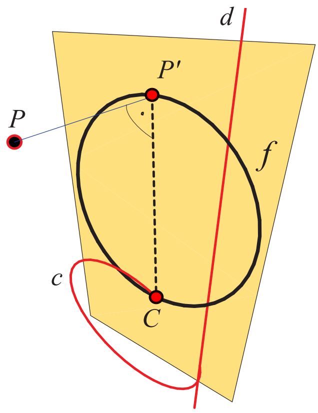


Figure 4

Now we can comprehend the generation of the pedal surfaces as it is shown in Fig. 5:

The pedal surface - type I is a simple infinite system of the circles f. The diameters of these circles are enclosed by the points on the ellipse c and the circle k (k is the circle in the plane through the pole P perpendicular to the line d. One of its diameters is enclosed by P and by the foot of P on the straight line d).

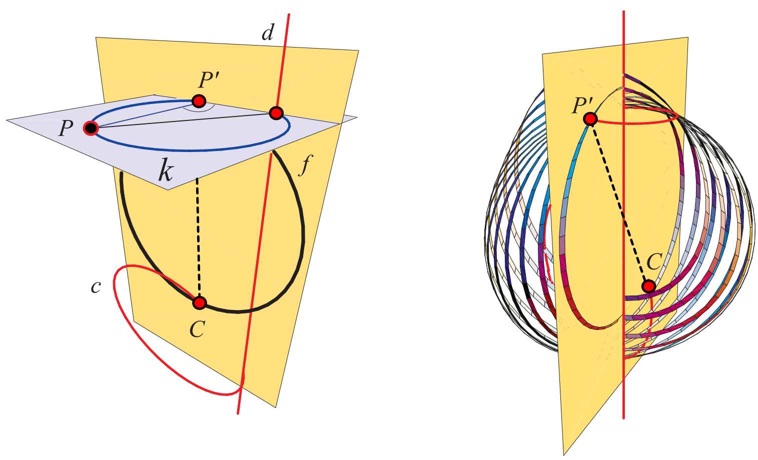


Figure 5

2.4 Parametric and homogeneous equations. Plotting.

The directed curves of the (1,2) congruence - type I are given with the equation (1), and the homogeneous coordinate of the finite pole P are

$$P(p:q:r:1), \quad p,q,r \in R. \tag{2}$$

According to the theorem 1 we can derive the following parametric equations of the pedal surfaces - type I

$$\begin{aligned} x(u,v) &= \mathrm{cos}u(R(u)\mathrm{sin}v + t_S(u)) \\ y(u,v) &= \mathrm{sin}u(R(u)\mathrm{sin}v + t_S(u)) \\ z(u,v) &= R(u)\mathrm{cos}v + z_S(u), \qquad u \in [-\frac{\pi}{2}, \frac{\pi}{2}],\ v \in [0, 2\pi], \end{aligned} \tag{3}$$

where $R(u)$, $t_S(u)$ and $z_S(u)$ depend on the real numbers a, b, c, A, B, p, q and r, $a > 0$, $ab^2 + c^2 \neq 0$. [2, p. 26-28]

Now we can define `ParF[a,b,c,A,B,p,q,r][u,v]` - the list of parametric equations and make parametric plots with *Mathematica* (see `pedals.nb`).

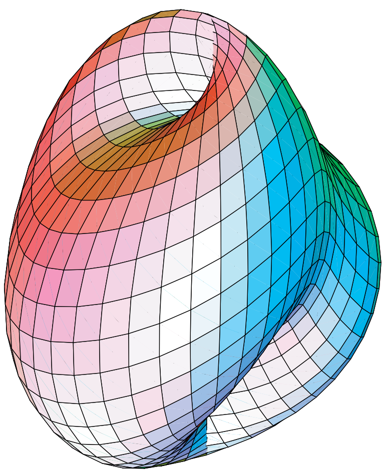


Figure 6: Parametric plot of `ParF[0.5,1,1,1,1,1,1,5][u,v]`.

We can also derive the homogeneous equation [2, p.31-32] of the surfaces

$$\begin{aligned} F(x,y,z,w) = & (x^2+y^2+z^2)(x^2+ay^2) \\ & +[(x^2+y^2-Axz-Byz)(bx+cy)-(px+qy+rz)(x^2+ay^2)]w \\ & +(bx+cy)(rAx+rBy-px-qy)w^2=0, \end{aligned} \tag{4}$$

and define the polynomial `F[a,b,c,A,B,p,q,r][x,y,z,w]` which we are going to use in the computation of singular points (see `pedals.nb`).

3. General properties of singularities on quartics

A point A is a m-fold *singular* point on an algebraic surface of the order n $(1 < m < n)$ if the tangents of the surface at the point A form m-order cone with the vertex A.

The highest singularity which a quartic can possess is a triple straight line and triple points.

• Multiple lines on quartics
- 1 triple straight line (ruled quartics)
- 1 double conic, 1 double straight line (ruled quartics)
- 1 double conic (this class contains cycloids)
- 3 double straight lines (Steiner's quartics)
- 2 double straight lines (ruled quartics)
- 1 double straight line (this class contains the pedal surfaces of (1,2) congruences)

• Kinds of double points
- *conical points* (tangent cones are quadric cones)

If the tangent cone is real, a conical point is *node*, and if it is imaginary the conical point is *isolated*.
- *binodes* (tangent cones degenerate into pairs of planes - biplanes)

If the biplanes are different and real, a binode is *ordinary*, if they are a pair of imaginary planes, a binode is *isolated*, and if the biplanes coincide, a binode is an *unode*.

• Multiple points on quartics with 1 double straight line
- triple points are always incident with a double straight line
- at most 2 triple points on a double line

- points on a double line are binodes
- at most 4 unodes on a double line (unodes on a double line is called *pinch-points*)
- pinch-points separate ordinary and isolated binodes on a double line
- at most 8 double points which are not incident with a double line

[7, p. 200-250], [6, p. 1633]

4. Singularities on the pedal surfaces of (1,2) congruences - type I

4.1 Analytical basis

The point $S(x_0, y_0, z_0)$ is a singular point on n-order algebraic surface which is given with n-degree algebraic equation $F(x, y, z) = 0$ if

$$\begin{aligned} &F(x_0, y_0, z_0) = 0 \qquad \text{and} \\ &F_x(x_0, y_0, z_0) = F_y(x_0, y_0, z_0) = F_z(x_0, y_0, z_0) = 0, \\ &\text{where } F_i = \frac{\partial F}{\partial i}, i \in \{x, y, z\}. \end{aligned} \tag{5}$$

Since the function $F(x, y, z)$ is n-degree polynomial, then it can be approximated by Taylor's polynomial in the neighbourhood of the double point $D(x_0, y_0, z_0)$, for which relation (5) is valid.

$$\begin{aligned} F(x, y, z) =& (x - x_0)^2 F_{xx}(D) + (y - y_0)^2 F_{yy}(D) + (z - z_0)^2 F_{zz}(D) \\ &+ 2(x - x_0)(y - y_0)F_{xy}(D) + 2(x - x_0)(z - z_0)F_{xz}(D) \\ &+ 2(y - y_0)(z - z_0)F_{yz}(D) + \mathbf{R}_3. \end{aligned} \tag{6}$$

If the quadratic form from (6) is equalized to 0, the equation of the tangent cone at the double point $D(x_0, y_0, z_0)$ is obtained.

Therefore, for the double point $D(x_0{:}y_0{:}z_0)$ it is enough to consider the matrix

$$\mathbf{A}(D) = \begin{pmatrix} F_{xx}(D) & F_{xy}(D) & F_{xz}(D) \\ F_{yx}(D) & F_{yy}(D) & F_{yz}(D) \\ F_{zx}(D) & F_{zy}(D) & F_{zz}(D) \end{pmatrix}. \tag{7}$$

Theorem 2 *Let $F(x,y,z)=0$ be the equation of n-order algebraic surface with a singular point D. Let $\mathbf{A}(D)$ be the matrix of quadratic form from Taylor's expansion of the function F in the neighbourhood of D. Then*

$Det\mathbf{A}(D)\neq 0$	$\Leftrightarrow$	*D is a conical point*
$Det\mathbf{A}(D)=0, rang\mathbf{A}(D)=2$	$\Leftrightarrow$	*D is a binode (ordinary or isolated)*
$Det\mathbf{A}(D)=0, rang\mathbf{A}(D)=1$	$\Leftrightarrow$	*D is an unode*
$\mathbf{A}(D)=\mathbf{0}$	$\Leftrightarrow$	*the multiplicity of D is >2.*

Proof. [2, p. 67].

4.2 Singularities on the pedal surfaces of (1,2) congruences - type I

With *Mathematica* we define the functions `AF[a,b,c,A,B,p,q,r][x0,y0,z0,w0]` and `BF[a,b,c,A,B,p,q,r][x0,y0,z0,w0]` for computing the matrix $\mathbf{A}(D)$ and the matrix $\mathbf{B}(D)$ for each singular point D on the surface given by the equation (4). The matrix $\mathbf{B}(D)$ is obtained from $\mathbf{A}(D)$ by cutting 3rd row and column (see `pedals.nb`).

In the following the pedal surfaces of (1,2) congruences - type I will be written Φ_{I}.

4.2.1 Singular points lying on a double straight line

Because the axis z is a double line of Φ_{I} it is clear that the points on it are binodes or triple points. For determination of their kinds it will be enough to consider the matrix $\mathbf{B}(D)$ and find the conditions for triple and pinch-points which separate ordinary and isolated binodes on a double line.

It is easy to prove that the point at infinity (0:0:1:0) is always an isolated binode of Φ_{I}, [2, p. 70].

1. Triple points

Theorem 3 *A point $T(0:0:t:1)$ is a triple point on the surface Φ_{I} if and only if*

- *$P=(0:0:r:1)$, then $T=P$* *or*
- *$P=(Ar:Br:r:1)$, when $T=O$.*

Proof. According to the theorem 2, $T(0{:}0{:}t{:}1)$ is a triple point of Φ_{I} if and only if $\mathbf{B}(T) = \mathbf{0}$.

Now we compute the matrix $\mathbf{B}(T)$

```
In[24]:= MatrixForm[BF[a,b,c,A,B,p,q,r][0,0,t,1]]
```

$$\mathtt{Out[24]} = \begin{pmatrix} 2b(-p+Ar)-2Abt-2rt+2t^2 & c(-p+Ar)+b(-q+Br)-bBt-Act \\ c(-p+Ar)+b(-q+Br)-bBt-Act & 2c(-q+Br)-2Bct-2art+2at^2 \end{pmatrix}$$

and obtain the conditions for a triple point

$$\begin{aligned} b(-p+Ar)-2Abt-2rt+2t^2 &= 0, \\ c(-p+Ar)+b(-q+Br)-bBt-Act &= 0, \\ c(-q+Br)-2Bct-2art+2at^2 &= 0. \end{aligned} \tag{8}$$

Now we apply the command `Reduce` on the equations (8).

```
In[25]:=Reduce[{b*(-p+A*r)-A*b*t-r*t+t^2==0,
               c*(-q+B*r)-B*c*t-a*r*t+a*t^2==0,
               c*(-p+A*r)+b*(-q+B*r)-b*B*t-A*c*t==0}]

Out[25]=
b==0 && c==0 && r==t ||
b==0 && c==0 && t==0 ||
p==0 && q==0 && r==t && t!=0 ||
p==A*r && q==B*r && t==0 && b!=0 ||
p==A*r && q==B*r && t==0 && c!=0 ||

a==-(c^2/b^2) && p==(A*b*r-A*b*t-r*t+t^2)/b &&
q==B*r-B*t+(c*r^3*t^3)/(b*r*t-b*t^2)^2-(3*c*r^2*t^4)/(b*r*t-b*t^2)^2
   +(3*c*r*t^5)/(b*r*t-b*t^2)^2-(c*t^6)/(b*r*t-b*t^2)^2 &&
b!=0 && r-t!=0 && t!=0
```

The equations generated by Reduce are equivalent to (8) and contain all the possible solutions. We obtained six equivalent systems but only the systems 3, 4 and 5 satisfy $ab^2 + c^2 \neq 0$ which is the condition of the congruence (1). □

Corollary 1 *The surface Φ_{I} has two different triple points if and only if $P \neq O$, $P \in d$ and $\gamma \perp d$. There are the points P and O.*

Proof. It is a direct consequence of the theorem 3. □

2. Pinch-points

Theorem 4 *A point $K(0:0:k:1)$ is a pinch-point on the surface Φ_{I} if and only if $\mathbf{B}(K) \neq \mathbf{0}$ and k is a null-point of the following polynomial*

$$\begin{aligned} P_F(z) \;=\;& 4az^4 - 4(aAb + Bc + 2ar)z^3 \\ & +(-b^2B^2 - A^2c^2 - 4cq + 8Bcr + 4ar^2 + 2b(ABc - 2ap + 4aAr))z^2 \\ & +2(b^2B(-q + Br) + b(Acq + 2apr - 2aAr^2 + Bc(p - 2Ar)) \\ & \quad +c(-Acp + A^2cr + 2r(q - Br)))z \\ & -(cp - bq + bBr - Acr)^2 \,. \end{aligned} \tag{9}$$

Proof. [2, p. 73]

Now, we can define the function `TriplePinch[a,b,c,A,B,p,q,r]` for computing the z-coordinates of triple and pinch points (see `pedals.nb`).

4.2.2 Singular points not lying on a double straight line

According to [2, p. 18,80-81] there are at most three double points on the surfaces Φ_{I}. They are the intersection points of the ellipse c and the circle k (Fig. 5).

Since the equations of the circle k are

$$x^2 + y^2 - px - qy = 0, \qquad z = r, \tag{10}$$

and the equations of the ellipse c are

$$x^2 + ay^2 + bx + cy = 0, \qquad Ax + By + z = 0, \tag{11}$$

then we can define the function `Double[a,b,c,A,B,p,q,r]` for computing the coordinates of double points not lying on the double line z (see `pedals.nb`).

5. The classification of the surfaces Φ_{I}

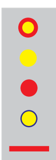
In [2, p. 75-83] the pedal surfaces Φ_{I} are classified according to the number of triple and pinch-points on a double line, and the number of conical points and binodes not lying on a double line. Sixteen types are obtained.

T - the number of triple points
P - the number of pinch-points
C - the number of conical points not lying on a double line
B - the number of binodes not lying on a double line

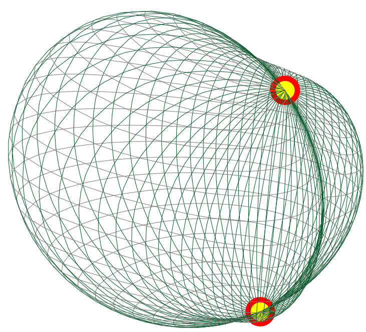
Type	**T**	**P**	**C**	**B**
I_1	2	0	0	0
I_2	1	0	0	0
I_3	1	2	0	0
I_4	1	1	0	0
I_5	0	4	0	0
I_6	0	3	0	0
I_7	0	2	0	0
I_8	0	1	0	0
I_9	0	2	3	0
I_{10}	0	2	1	1
I_{11}	0	2	2	0
I_{12}	0	2	0	1
I_{13}	0	2	1	0
I_{14}	0	1	2	0
I_{15}	0	1	0	1
I_{16}	0	1	1	0

The plot-functions (which enable us to point out singular points) are defined in the program *Mathematica*, and for each type of the surface one example is drawn (see `pedals.nb`).

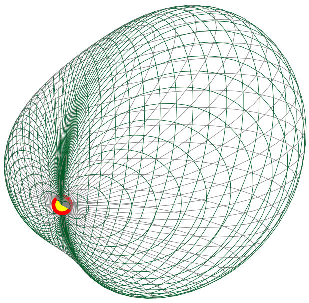
Marks



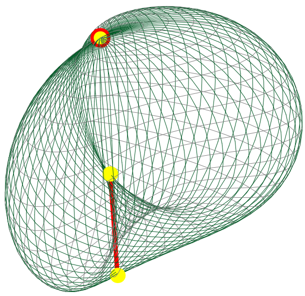
- a triple point
- a pinch point
- a conical point
- a binode not lying on a double line
- ordinary binodes on a double line



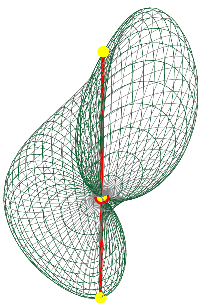
Type $\mathbf{I_1}$: F[3,-2,2,0,0,0,0,2], $T_1(0{:}0{:}0{:}1)$, $T_2(0{:}0{:}2{:}1)$



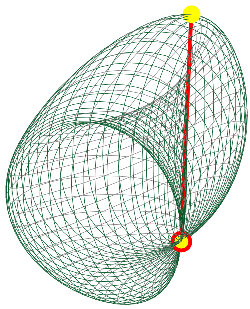
Type $\mathbf{I_2}$: F[1/2,-2,2,0,0,0,0,0],
T(0:0:0:1)



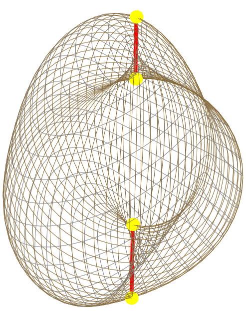
Type $\mathbf{I_{3,1}}$: F[1,2,4,1,-1,0,0,8],
T(0:0:8:1), P_1(0:0:-4.16:1), P_2(0:0:2.16:1)



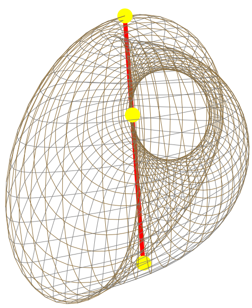
Type $\mathbf{I_{3,2}}$: F[1,-1,1,1,1,0,0,0],
T(0:0:0:1), P_1(0:0:-1:1), P_2(0:0:1:1)



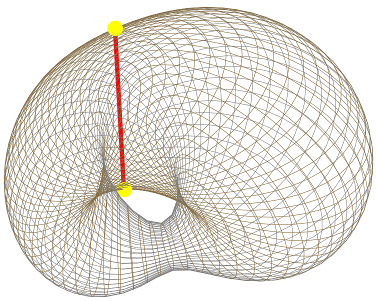
Type $\mathbf{I_4}$: F[1,-1,1,1,1,1,1,1],
T(0:0:0:1), P(0:0:2:1)



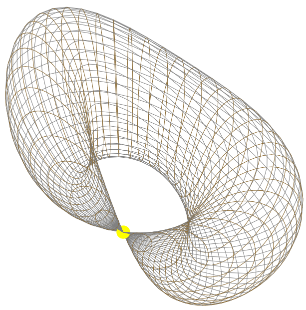
Type $\mathbf{I_5}$: F[0.5,1,1,1,1,1,1,5], P_1(0:0:0:1)
P_2(0:0:2:1),P_3(0:0:5:1),P_4(0:0:6:1)



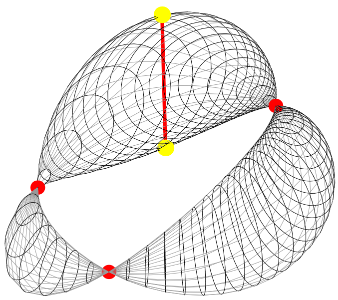
Type $\mathbf{I_6}$: F[$3+2\sqrt{2}$,1,-1,-1,-1,1,1,1],
P_1(0:0:-$\sqrt{2}$:1), P_2(0:0:2-$\sqrt{2}$:1), P_3(0:0:$\sqrt{2}$:1)



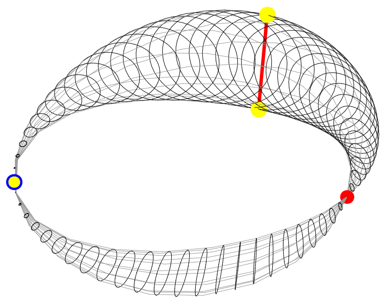
Type $\mathbf{I_7}$: F[2,-2,2,-0.25,-0.25,2,2,1],
P_1(0:0:-1.17:1), P_2(0:0:1.92:1)



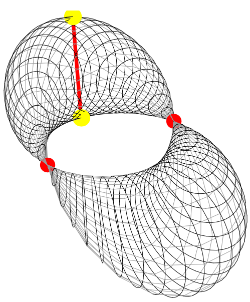
Type $\mathbf{I_8}$: F[1,-2,0,0,-1,1,0,0],
P(0:0:0:1)



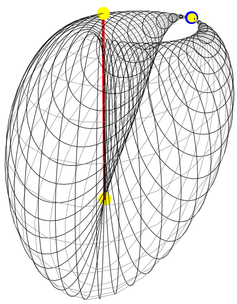
Type $\mathbf{I_9}$: F[6,-1,-4,0,0,1,1,0],
P_1(0:0:-0.448:1), P_2(0:0:0.448:1),
C_1(1.2:0.6:0:1), C_2(1:0:0:1), C_3(-0.2:0.6:0:1)



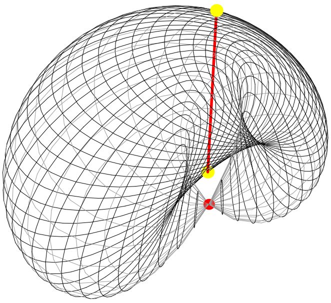
Type $\mathbf{I_{10}}$: F[2,1,2,0,0,-0.25(3+$\sqrt{3}$),-1,0],
P_1(0:0:-0.319:1), P_2(0:0:0.319:1),
C(0:-1:0:1), B(-0.5(1+$\sqrt{3}$):-0.5:0:1)



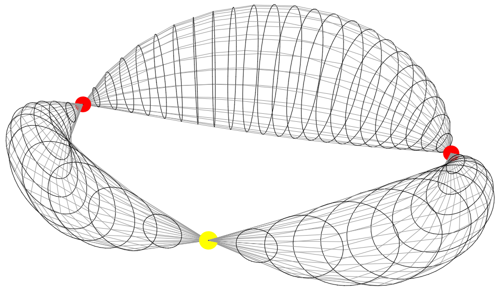
Type $\mathbf{I_{11}}$: F[1,1,1,-1,-1,-1,-1,-1], P_1(0:0:0:1),
P_2(0:0:-1:1), C_1(-1:0:-1:1), C_2(0:-1:-1:1)



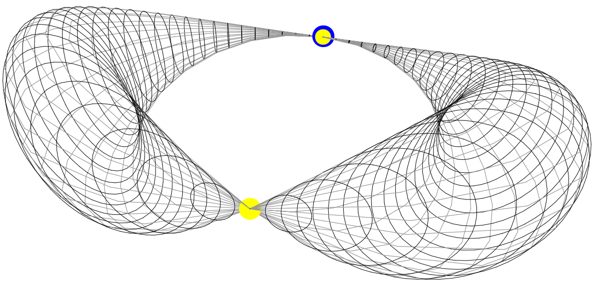
Type $\mathbf{I_{12}}$: F[1,-2,0,0,-2,2,0,2],
P_1(0:0:-1.509:1), P_2(0:0:2:1), B(1:1:2:1)



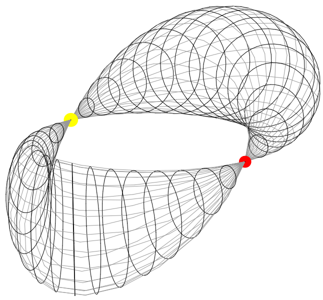
Type $\mathbf{I_{13}}$: F[2,1,1,1,1,-1,1,1],
P_1(0:0:-0.173:1), P_2(0:0:1.579:1), C(-1:0:1:1)



Type $\mathbf{I_{14}}$: F[0.25,-2.5,0,0,0,4,0,0],
P(0:0:0:1), C_1(2:2:0:1),C_2(2:-2:0:1)



Type $\mathbf{I_{15}}$: F[0.25,-2,0,0,0,2,0,0],
P(0:0:0:1), B(2:0:0:1)



Type $\mathbf{I_{16}}$: F[1,-2,0,0,-1,2,0,0],
P(0:0:0:1), C(2:0:0:1)

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