VISUALIZATION OF SPECIAL CIRCULAR SURFACES

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ABSTRACT: In this paper we construct a new class of algebraic surfaces in three-dimensional Euclidean space generated by a cyclic-harmonic curve and a congruence of circles. We study their algebraic properties and visualize them with the program *Mathematica*.

Keywords: circular surface, cyclic-harmonic curve, singular point, congruence of circles

1. INTRODUCTION

In article [5] a *circular surface* is defined as (the image of) a map $V: I \times \mathbb{R}/2\pi\mathbb{Z} \longrightarrow \mathbb{R}^3$ given by

$$V(t,\theta) = \gamma(t) + r(t)(\cos\theta\,\mathbf{a}_1(t) + \sin\theta\,\mathbf{a}_2(t)), \quad (1)$$

where $\gamma, \mathbf{a}_1, \mathbf{a}_2 : I \longrightarrow \mathbb{R}^3$ and $r: I \longrightarrow \mathbb{R}_{>0}$.
It is assumed that $\langle \mathbf{a}_1, \mathbf{a}_1 \rangle = \langle \mathbf{a}_2, \mathbf{a}_2 \rangle = 1$ and $\langle \mathbf{a}_1, \mathbf{a}_2 \rangle = 0$ for all $t \in I$, where $\langle , \rangle$ denotes the canonical inner product on $\mathbb{R}^3$. The curve γ is called a *base curve* and the pair of curves $\mathbf{a}_1$, $\mathbf{a}_2$ is said to be a *director frame*. The standard circles $\theta \mapsto r(t)(\cos\theta\mathbf{a}_1(t) + \sin\theta\mathbf{a}_2(t))$ are called *generating circles*. Furthermore, in this paper the authors classify and investigate the differential properties of this one-parameter family of standard circles with a fixed radius.

In paper [2] the authors defined three types (elliptic, parabolic and hyperbolic) of congruences $\mathscr{C}(p)$ that contain circles passing through two points P_1 and P_2. Then for a given congruence $\mathscr{C}(p)$ of a certain type and a given curve α they defined a circular surface $\mathscr{CS}(\alpha, p)$ as the system of circles from $\mathscr{C}(p)$ that intersect α. These surfaces contain the generating circles of variable radii.

The *rose surfaces* studied in [1] are circular surfaces $\mathscr{CS}(\alpha, p)$ where α is a rose (rhodonea curve), see [6], and $\mathscr{C}(p)$ is an elliptic or parabolic congruence. The rose lies in the plane perpendicular to the line P_1P_2 having the directing point P_1 as its multiple point.

In this paper we extend α to all cyclic-harmonic curves $\rho = \cos\frac{n}{d}\varphi + a$, $a \in \mathbb{R}^+ \cup \{0\}$ (see [4], [7]), and include a hyperbolic congruence $\mathscr{C}(p)$. For such circular surfaces $\mathscr{CS}(\alpha, p)$ we give the overview of their algebraic properties and visualize their numerous forms with the program *Mathematica*.

2. CYCLIC-HARMONIC CURVES

A *cyclic-harmonic curve* is given by the following polar equation

$$r(\varphi) = b\cos\frac{n}{d}\varphi + a, \ \varphi \in [0, 2d\pi], \quad (2)$$

where $\frac{n}{d}$ is a positive rational number in lowest terms and $a, b \in \mathbb{R}^+$. It is a locus of a composition of two simultaneusly motions: a *simple harmonic* motion $r(\varphi) = b\cos nt + a$, and an *uniform angular* motion $\varphi = dt$.

According to [7], a cyclic-harmonic curve is called *foliate*, *prolate*, *cuspitate* or *curtate*, if $a = 0$, $a < b$, $a = b$ or $a > b$, respectively. Without the loss of generality, we will suppose that $b = 1$ and denote a cyclic-harmonic curve by $CH(n,d,a)$. Some exaples of these curves are shown in Figure 1.

The curves $CH(n,d,0)$ are called *roses* (*rhodonea*) and have been treated in [1] where the author obtained their implicit equation (3). The reader is invited to read the paper and see the method of deriving because in the same way we derived implicit equations (4) and (5) that are the equations of $CH(n,d,a)$ for all other cases.

Roses or rhodonea curves $CH(n,d,0)$ are given by the following algebraic equation:

$$\left(\sum_{k=0}^{\lfloor \frac{d}{2}\rfloor}\sum_{j=0}^{k}(-1)^{j+k}\binom{d}{2k}\binom{k}{j}(x^2+y^2)^{\frac{n+d}{2}-k+j}\right)^s-\left(\sum_{i=0}^{\lfloor \frac{d}{2}\rfloor}(-1)^i\binom{n}{2i}x^{n-2i}y^{2i}\right)^s=0, \tag{3}$$

where $s=1$ if $n\cdot d$ is odd and $s=2$ if $n\cdot d$ is even.

If $a\neq 0$ and n is an even number (d must be odd), the algebrac equation of $CH(n,d,a)$ is the following:

$$(x^2+y^2)^{n+1}\Big(\sum_{j=0}^{\lfloor \frac{d}{2}\rfloor}\sum_{k=0}^{j}\sum_{l=0}^{\lfloor \frac{d-2k}{2}\rfloor}(-1)^{d-k-1}\binom{d}{2j}\binom{j}{k}\binom{d-2k}{2l+1}a^{d-2k-2l-1}(x^2+y^2)^l\Big)^2 \tag{4}$$
$$=\Big(\sum_{i=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^i\binom{n}{2i}x^{n-2i}y^{2i}-(x^2+y^2)^{\frac{n}{2}}\sum_{j=0}^{\lfloor \frac{d}{2}\rfloor}\sum_{k=0}^{j}\sum_{l=0}^{\lfloor \frac{d-2k}{2}\rfloor}(-1)^{d-k}\binom{d}{2j}\binom{j}{k}\binom{d-2k}{2l}a^{d-2k-2l}(x^2+y^2)^l\Big)^2.$$

If $a\neq 0$ and n is an odd number, the algebrac equation of $CH(n,d,a)$ is the following:

$$(x^2+y^2)^{n}\Big(\sum_{j=0}^{\lfloor \frac{d}{2}\rfloor}\sum_{k=0}^{j}\sum_{l=0}^{\lfloor \frac{d-2k}{2}\rfloor}(-1)^{d-k}\binom{d}{2j}\binom{j}{k}\binom{d-2k}{2l}a^{d-2k-2l}(x^2+y^2)^l\Big)^2 \tag{5}$$
$$=\Big(\sum_{i=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^i\binom{n}{2i}x^{n-2i}y^{2i}-(x^2+y^2)^{\frac{n+1}{2}}\sum_{j=0}^{\lfloor \frac{d}{2}\rfloor}\sum_{k=0}^{j}\sum_{l=0}^{\mathrm{D}}(-1)^{d-k-1}\binom{d}{2j}\binom{j}{k}\binom{d-2k}{2l+1}a^{d-2k-2l-1}(x^2+y^2)^l\Big)^2,$$

where

$$\mathrm{D}=\begin{cases}\lfloor\frac{d-2k}{2}\rfloor & \text{if } d \text{ is an odd number,}\\ \lfloor\frac{d-2k}{2}\rfloor-1 & \text{if } d \text{ is an even number.}\end{cases}$$

Since equations (4) and (5) are of the degree $2(n+d)$, $CH(n,d,a)$ is $2(n+d)$ order curve. According to [3, p.251], the origin $O(0,0)$ is an $2n$-fold point of these curves, while the absolute points A_1, A_2 are $2d$-fold or $(n+d)$-fold when $d<n$ or $n<d$, respectively (see [8]). These algebraic properties of $CH(n,d,a)$ are given in Table 1. Since $CH(n,d,a)$ is unicursal [4], besides O, A_1, A_2 it possesses another multiple points.

Table 1: Properties of $CH(n,d,a)$

		order	**O**	$\mathbf{A_1, A_2}$
$a=0$	$\mathbf{d<n}$	$n+d$	n-fold	d-fold
$n\cdot d$ odd	$\mathbf{d>n}$	$n+d$	n-fold	$\frac{n+d}{2}$-fold
other	$\mathbf{d<n}$	$2(n+d)$	$2n$-fold	$2d$-fold
cases	$\mathbf{d>n}$	$2(n+d)$	$2n$-fold	$(n+d)$-fold

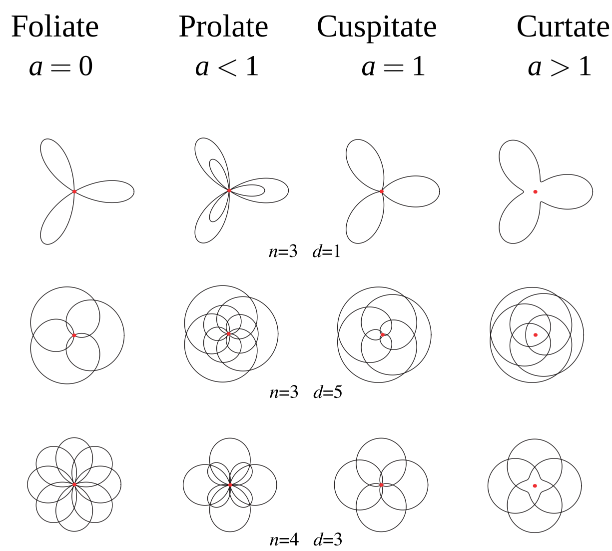

Figure 1: Some examples of $CH(n,d,a)$

3. GENERALIZED ROSE-SURFACES

Let $\mathscr{C}(p)$ be a congruence of circles that consists of circles in Euclidean space $\mathbb{E}^3$ passing through two given points P_1, P_2. The points P_1, P_2 lie on the axis z and are given by the coordinates $(0,0,\pm p)$, where $p=\sqrt{q}$, $q\in\mathbb{R}$. If q is greater, equal or less than zero, $\mathscr{C}(p)$ is an elliptic, parabolic or hyperbolic congruence, respectively.

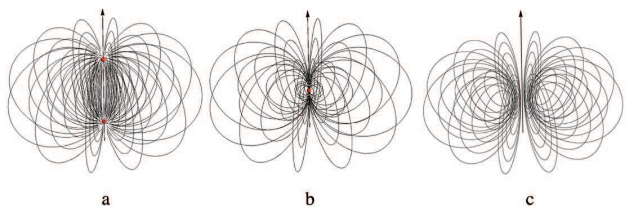

Figure 2: Some circles of an elliptic, parabolic or hyperbolic congruence $\mathscr{C}(p)$ are shown in a, b and c, respectively.

For every point A ($A\notin z$), there exists a unique circle $c^A(p)\in\mathscr{C}(p)$ passing through the points A, P_1 and P_2. A point is a singular point of a congruence if infinitely many curves pass through it. The singular points of $\mathscr{C}(p)$ are the points on the axis z and the absolute points of $\mathbb{E}^3$.

For a given congruence $\mathscr{C}(p)$ and a given curve α a *circular surface* $\mathscr{CS}(\alpha,p)$ is defined as the system of circles from $\mathscr{C}(p)$ that intersect the curve α (see Figure 3).

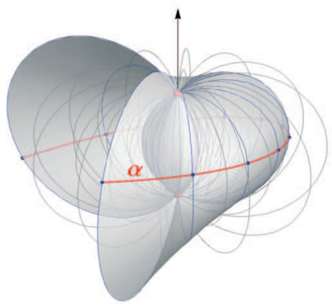

Figure 3: The circular surface $\mathscr{CS}(\alpha,p)$.

In [2] the authors have shown that if α is an m^{th} order algebraic curve that cuts the axis z at z' points, the absolute conic at a' pairs of the absolute points and with the points P_1 and P_2 as p_1'-fold and p_2'-fold points, respectively, then, the following statements hold:

1. $\mathscr{CS}(\alpha,p)$ is an algebraic surfaces of the order $3m-(z'+2a'+2p_1'+2p_2')$.
2. The absolute conic is an $m-(z'+p_1'+p_2')$-fold curve of $\mathscr{CS}(\alpha,p)$.
3. The axis z is an $(m-2a'+z')$–fold line of $\mathscr{CS}(\alpha,p)$.
4. The points P_1, P_2 are $2m-(2a'+p_1'+p_2')$-fold points of $\mathscr{CS}(\alpha,p)$.

Definition 1 *A generalized rose-surface is a circular surface $\mathscr{CS}(\alpha,p)$ where the curve α is a cyclic-harmonic curve $CH(n,d,a)$.*

The generalized rose surfaces $\mathscr{CS}(\alpha,p)$ for $\alpha=CH(n,d,0)$ and $p\in\mathbb{R}$ are simply called rose-surfaces and were detail studied in [1]. This paper presents the extension of these surfaces since we are also dealing with $\alpha=CH(n,d,a)$ for $a\neq 0$ and $p\in\mathbb{C}$. In this way numerous forms of a new class of surfaces with nice visualizations and interesting algebraic properties can be obtained.

Let $CH(n,d,a)$ lay in any plane of $\mathbb{E}^3$, let its $2n$-fold (n-fold) point be denoted by O, and let its another real j-fold points be denoted by D^j. According to the properties 1. - 4. of $\mathscr{CS}(\alpha,p)$ given above, and the properties of $CH(n,d,a)$ given in Table 1, we can made a preliminary classification of generalized rose-surfaces according to their order and the multiplicity of the axes z, the absolute points and the points P_1, P_2. Since this classification depends on the position of $CH(n,d,a)$ with respect to the singular points of the congruence $\mathscr{C}(p)$, the following cases should be taken into consideration:

TYPE 1: $O=P_1$ or $O=P_2$;
TYPE 2: $O\in z$, $O\neq P_i$, $i=1,2$;
TYPE 3: $D^j=P_1$ or $D^j=P_2$;
TYPE 4: $D^j\in z$, $D^j\neq P_i$, $i=1,2$;
TYPE 5: $\forall Z\in z$, $Z\notin CH(n,d,a)$.

The authors suppose that it is worth to study a complete class of these surfaces in a further work. Here, we give only some examples.

4. VISUALIZATION OF GENERALIZED ROSE-SURFACES

In [2] we derived the parametric equations of circular surfaces $\mathscr{CS}(\alpha, p)$ that enable *Mathematica* visualizations of any generalized rose-surface. Some examples are shown in the following figures.

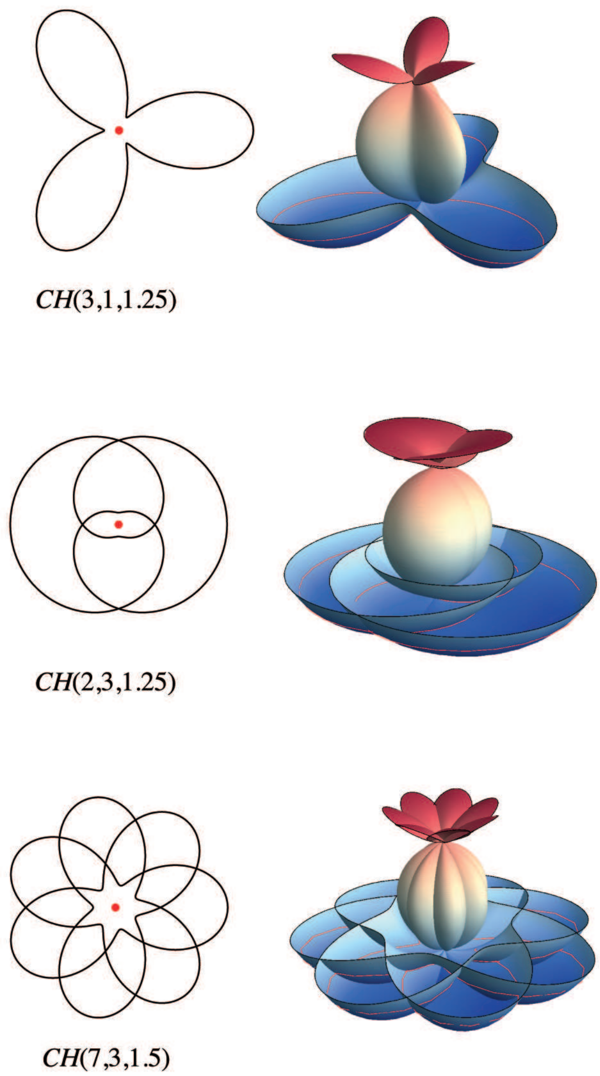

Figure 4: Three surfaces of type 1. The parameter $p = 1$, the curve CH lies in the plane $z = -1$ and $O = P_1$.

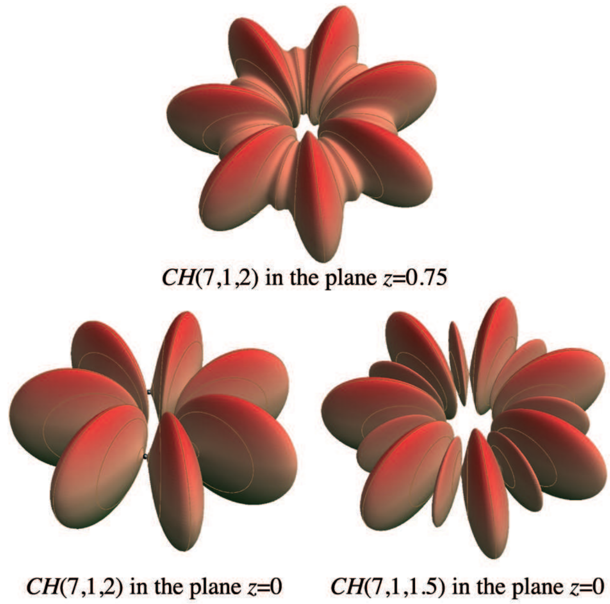

Figure 5: Three surfaces of type 2. The point O lies on the axis z, $O \neq P_1, P_2$, and $p = i$.

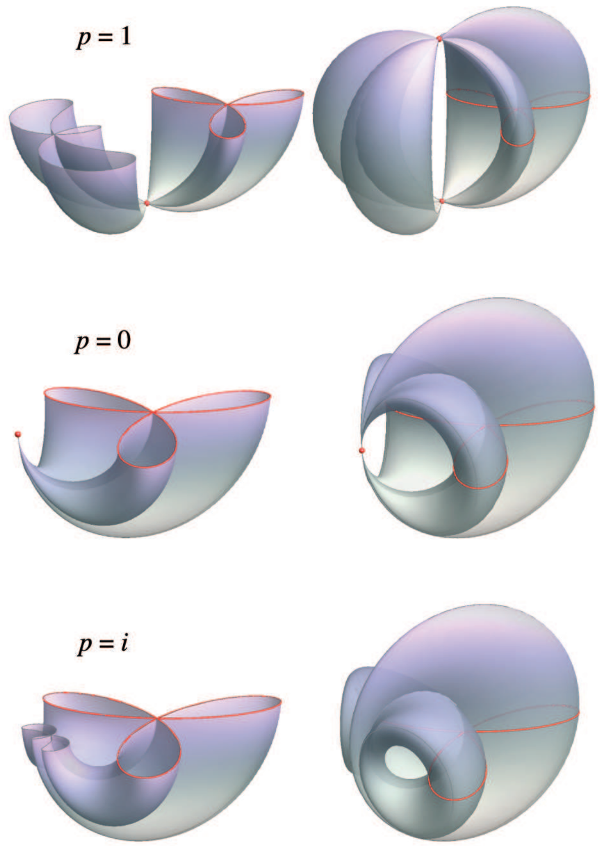

Figure 6: Three surfaces of type 5. $CH(3,1,0)$ lies in the plane $z = 0$, $O = (1,0,0)$.

5. CONCLUSIONS

For a given congruence of circles $\mathscr{C}(p)$ and a given curve α a circular surface $\mathscr{CS}(\alpha, p)$ is defined as the system of circles from $\mathscr{C}(p)$ that intersect α. In this paper the authors define and visualize a special class of $\mathscr{CS}(\alpha, p)$ when α is a cyclic-harmonic curve. The detail studying of a complete class of such surfaces is planned to be the subject in the authors' further work.

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