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GENERALIZED ROSE SURFACES END THEIR VISUALIZATIONS

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Abstract. In this paper we construct a new class of algebraic surfaces in three-dimensional Euclidean space generated by a cyclic-harmonic curve and a congruence of circles. We study their properties and visualize them with the program *Mathematica*.

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1. Introduction

The study of surfaces with some special properties in $\mathbb{R}^3$ takes one of the most important places in the classical geometry. Such surfaces are ruled surfaces, a one-parameter families of straight lines. Their value is visible in many areas of science and engineering like architecture, civil engineering, mechanical engineering and CAD. In paper [5] the authors defined and studied a one-parameter family of circles which they named *circular surface*. Assured in their applicability, the authors investigated their properties by using methods of differential geometry.

In [5] the definition of a *circular surface* is given as follows:

Definition 1.1. A *circular surface* is (the image of) a map $V\colon I\times\mathbb{R}/2\pi\mathbb{Z}\to\mathbb{R}^3$ defined by

$$V(t,\theta)=\gamma(t)+r(t)(\cos\theta\mathbf{a}_1(t)+\sin\theta\mathbf{a}_2(t)),$$

where $\gamma,\mathbf{a}_1,\mathbf{a}_2\colon I\to\mathbb{R}^3$ and $r\colon I\to\mathbb{R}_{>0}$.

It is assumed that $\langle\mathbf{a}_1,\mathbf{a}_1\rangle=\langle\mathbf{a}_2,\mathbf{a}_2\rangle=1$ and $\langle\mathbf{a}_1,\mathbf{a}_2\rangle=0$ for all $t\in I$, where $\langle,\rangle$ denotes the canonical inner product on R^3. The curve γ is called a *base curve* and the pair of curves $\mathbf{a}_1$, $\mathbf{a}_2$ is said to be a *director frame*. The standard circles $\theta\mapsto r(t)(\cos\theta\mathbf{a}_1(t)+\sin\theta\mathbf{a}_2(t))$ are called *generating circles*. In [5] the authors classify and investigate the differential properties of this one-parameter family of standard circles with a fixed radius.

In paper [2] the authors defined three types (elliptic, parabolic and hyperbolic) of congruences $\mathcal{C}(p)$ that contain circles passing through two points P_1 and P_2. Then, for a given congruence $\mathcal{C}(p)$ of a certain type and a given curve

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α they defined a circular surface $\mathcal{CS}(\alpha, p)$ as the system of circles from $\mathcal{C}(p)$ that intersect α. These surfaces contain the generating circles of variable radii.

The *rose surfaces* studied in [1] are circular surfaces $\mathcal{CS}(\alpha, p)$ where α is a rose (rhodonea curve) given by the polar equation $\rho = \cos \frac{n}{d}\varphi$, and $\mathcal{C}(p)$ is an elliptic or parabolic congruence. The rose lies in the plane perpendicular to the axis z having the directing point P_i as its multiple point.

In this paper we extend α to all cyclic-harmonic curves $\rho = \cos \frac{n}{d}\varphi + a$, $a \in \mathbb{R}^+ \cup \{0\}$ (see [4], [6]), and include a hyperbolic congruence $\mathcal{C}(p)$. Such circular surfaces $\mathcal{CS}(\alpha, p)$ are very interesting from the engineering point of view since on a surface lie the curves which are the locus of the composition of three simultaneous motions: two simple harmonic motions and an uniform motion. We believe that also the other properties of the surfaces are worth of study. So, we give an overview of their algebraic properties and visualize their numerous forms with the program *Mathematica*.

2. Cyclic-Harmonic Curve

A *cyclic-harmonic curve* is given by the following polar equation

$$r(\varphi) = b\cos\frac{n}{d}\varphi + a,\ \varphi \in [0, 2d\pi], \tag{2.1}$$

where $\frac{n}{d}$ is a positive rational number in lowest terms and $a, b \in \mathbb{R}^+$. It is a locus of a composition of two simultaneous motions: a *simple harmonic* motion $r(\varphi) = b\cos nt + a$, and an *uniform angular* motion $\varphi = dt$.

According to [6], a cyclic-harmonic curve is called *foliate*, *prolate*, *cuspitate* or *curtate* if $a = 0$, $a < b$, $a = b$ or $a > b$, respectively. Without the loss of generality, we will suppose that $b = 1$ and denote a cyclic-harmonic curve by $CH(n, d, a)$. Some examples of these curves are shown in Figure 1.

The curves $CH(n, d, 0)$ are called *roses* or *rhodonea curves* and have been treated in [1]. The reader is invited to read the paper and find more about their algebraic properties. Here we give only their implicit equation:

$$\left(\sum_{k=0}^{\lfloor \frac{d}{2} \rfloor}\sum_{j=0}^{k}(-1)^{j+k}\binom{d}{2k}\binom{k}{j}(x^2+y^2)^{\frac{n+d}{2}-k+j}\right)^s - \left(\sum_{i=0}^{\lfloor \frac{d}{2} \rfloor}(-1)^i\binom{n}{2i}x^{n-2i}y^{2i}\right)^s = 0, \tag{2.2}$$

where $s = 1$ if $n \cdot d$ is odd and $s = 2$ if $n \cdot d$ is even.

Theorem 2.1. *For a cyclic-harmonic curve $CH(n, d, a)$, Table 1 is valid:*

$CH(n,d,a)$		order of $CH(n,d,a)$	multiplicity of the origin O	multiplicity of the absolute points
$a = 0$	$d < n$	$n + d$	n	d
$n \cdot d$ odd	$d > n$	$n + d$	n	$\frac{n+d}{2}$
other	$d < n$	$2(n + d)$	$2n$	$2d$
cases	$d > n$	$2(n + d)$	$2n$	$(n + d)$

Table 1: Properties of $CH(n, d, a)$

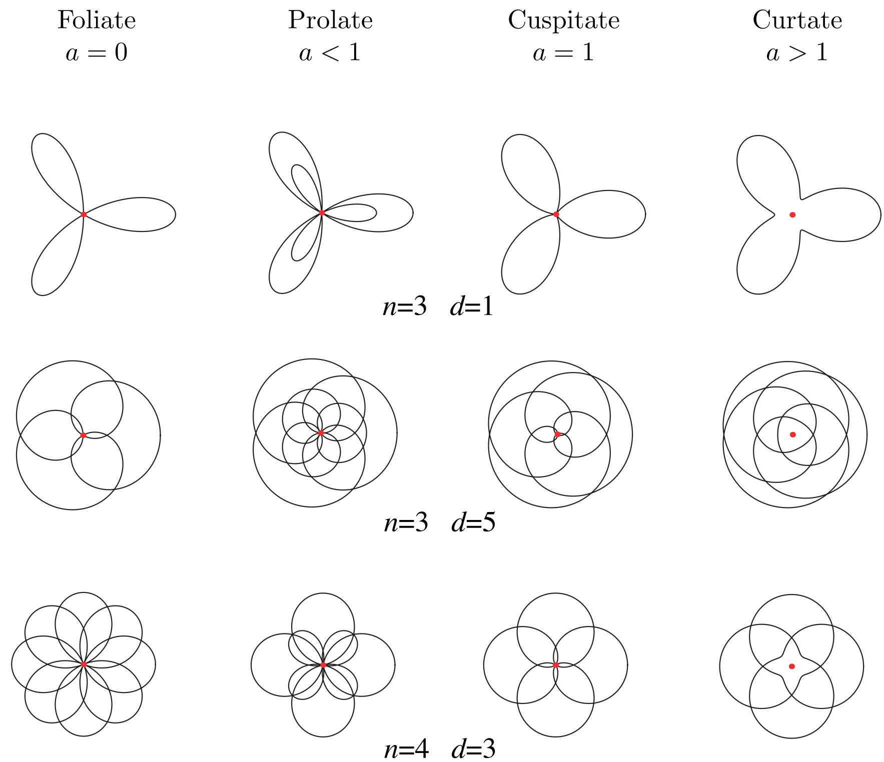


Figure 1: Some examples of $CH(n, d, a)$

Proof. Statements regarding roses were presented in [1]. So only the cases for $a \neq 0$ are left to be proved.

By substituting the standard formulas for conversion from the polar to the Cartesian coordinates $r(\varphi) = \sqrt{x^2 + y^2}$, $\cos\varphi = \frac{x}{\sqrt{x^2+y^2}}$, $\sin\varphi = \frac{y}{\sqrt{x^2+y^2}}$ into (2.1), we obtain the following condition for the points on $CH(n, d, a)$:

$$\cos\frac{n}{d}\varphi = \sqrt{x^2 + y^2} - a, \quad \sin\frac{n}{d}\varphi = \sqrt{1 - (\sqrt{x^2 + y^2} - a)^2}. \tag{2.3}$$

Since $\cos d\frac{n}{d}\varphi = \cos n\varphi$, the cosine multiple angle formula $\cos n\varphi = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n}{2i} (\sin\varphi)^{2i} (\cos\varphi)^{n-2i}$ gives us the following equality:

$$\sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} (-1)^j \binom{d}{2j} (\sin\frac{n}{d}\varphi)^{2j} (\cos\frac{n}{d}\varphi)^{d-2j} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n}{2i} (\sin\varphi)^{2i} (\cos\varphi)^{n-2i}. \tag{2.4}$$

Thus,

$$(x^2+y^2)^{\frac{n}{2}}\sum_{j=0}^{\lfloor\frac{d}{2}\rfloor}(-1)^j\binom{d}{2j}(1-(\sqrt{x^2+y^2}-a)^2)^j(\sqrt{x^2+y^2}-a)^{d-2j}$$
$$=\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^i\binom{n}{2i}x^{n-2i}y^{2i}. \tag{2.5}$$

After applying the binomial theorem to the expression $(1-(\sqrt{x^2+y^2}-a)^2)^j$, equation (2.5) turns into

$$(x^2+y^2)^{\frac{n}{2}}\sum_{j=0}^{\lfloor\frac{d}{2}\rfloor}\sum_{k=0}^{j}(-1)^{2j-k}\binom{d}{2j}\binom{j}{k}(\sqrt{x^2+y^2}-a)^{d-2k}=\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^i\binom{n}{2i}x^{n-2i}y^{2i}. \tag{2.6}$$

Using the same formula again, this time applying it to $(\sqrt{x^2+y^2}-a)^{d-2k}$, the equation above becomes:

$$(x^2+y^2)^{\frac{n}{2}}\sum_{j=0}^{\lfloor\frac{d}{2}\rfloor}\sum_{k=0}^{j}\sum_{l=0}^{d-2k}(-1)^{d-k-l}\binom{d}{2j}\binom{j}{k}\binom{d-2k}{l}a^{d-2k-l}(x^2+y^2)^{\frac{l}{2}}$$
$$=\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^i\binom{n}{2i}x^{n-2i}y^{2i}. \tag{2.7}$$

The left side of equation (2.7) can be written in the following form:

$$(x^2+y^2)^{\frac{n}{2}}\sum_{j=0}^{\lfloor\frac{d}{2}\rfloor}\sum_{k=0}^{j}\sum_{l=0}^{\lfloor\frac{d-2k}{2}\rfloor}(-1)^{d-k}\binom{d}{2j}\binom{j}{k}\binom{d-2k}{2l}a^{d-2k-2l}(x^2+y^2)^{l}$$
$$+(x^2+y^2)^{\frac{n+1}{2}}\sum_{j=0}^{\lfloor\frac{d}{2}\rfloor}\sum_{k=0}^{j}\sum_{l=0}^{\mathrm{D}}(-1)^{d-k-1}\binom{d}{2j}\binom{j}{k}\binom{d-2k}{2l+1}a^{d-2k-2l-1}(x^2+y^2)^{l}, \tag{2.8}$$

where

$$\mathrm{D}=\begin{cases}\lfloor\frac{d-2k}{2}\rfloor & \text{if } d \text{ is an odd number,}\\ \lfloor\frac{d-2k}{2}\rfloor-1 & \text{if } d \text{ is an even number.}\end{cases} \tag{2.9}$$

Therefore, the implicit equations of cyclic-harmonic curves are:

– if n is an even number (d must be odd)

$$(x^2+y^2)^{n+1}\Big(\sum_{j=0}^{\lfloor\frac{d}{2}\rfloor}\sum_{k=0}^{j}\sum_{l=0}^{\lfloor\frac{d-2k}{2}\rfloor}(-1)^{d-k-1}\binom{d}{2j}\binom{j}{k}\binom{d-2k}{2l+1}a^{d-2k-2l-1}(x^2+y^2)^l\Big)^2$$

(2.10)

$$=\Big(\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^i\binom{n}{2i}x^{n-2i}y^{2i}$$
$$-(x^2+y^2)^{\frac{n}{2}}\sum_{j=0}^{\lfloor\frac{d}{2}\rfloor}\sum_{k=0}^{j}\sum_{l=0}^{\lfloor\frac{d-2k}{2}\rfloor}(-1)^{d-k}\binom{d}{2j}\binom{j}{k}\binom{d-2k}{2l}a^{d-2k-2l}(x^2+y^2)^l\Big)^2$$

– if n is an odd number

$$(x^2+y^2)^{n}\Big(\sum_{j=0}^{\lfloor\frac{d}{2}\rfloor}\sum_{k=0}^{j}\sum_{l=0}^{\lfloor\frac{d-2k}{2}\rfloor}(-1)^{d-k}\binom{d}{2j}\binom{j}{k}\binom{d-2k}{2l}a^{d-2k-2l}(x^2+y^2)^l\Big)^2$$

(2.11)

$$=\Big(\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^i\binom{n}{2i}x^{n-2i}y^{2i}$$
$$-(x^2+y^2)^{\frac{n+1}{2}}\sum_{j=0}^{\lfloor\frac{d}{2}\rfloor}\sum_{k=0}^{j}\sum_{l=0}^{\mathrm{D}}(-1)^{d-k-1}\binom{d}{2j}\binom{j}{k}\binom{d-2k}{2l+1}a^{d-2k-2l-1}(x^2+y^2)^l\Big)^2$$

where D is given by (2.9).

Since both equations (2.10) and (2.11) are of the degree $2(n+d)$, the algebraic curve $CH(n,d,a)$ is of the order $2(n+d)$.

According to [3, p.251], for an nth order algebraic plane curve which passes through the origin $O(0,0)$ and is given by an algebraic equation $P^n(x,y)=0$, the tangent lines at the origin are given by a homogeneous algebraic equation $f^k(x,y)=0$, where $f^k(x,y)$ is the term of $P^n(x,y)$ with the lowest degree. Thus, $O(0,0)$ is an $2n$-fold point of $CH(n,d,a)$ because the tangent lines are given by the following algebraic equations of degree $2n$:

– if n is an even number (d must be odd)

$$\Big(\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^i\binom{n}{2i}x^{n-2i}y^{2i}-\mathrm{A}\cdot(x^2+y^2)^{\frac{n}{2}}\Big)^2=0 \tag{2.12}$$

– if n is an odd number

$$\mathrm{A}^2\cdot(x^2+y^2)^n-\Big(\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^i\binom{n}{2i}x^{n-2i}y^{2i}\Big)^2=0 \tag{2.13}$$

where

$$\mathrm{A}=\sum_{j=0}^{\lfloor\frac{d}{2}\rfloor}\sum_{k=0}^{j}(-1)^{d-k}\binom{d}{2j}\binom{j}{k}a^{d-2k}=\frac{1}{2}\Big(\Big(-\sqrt{a^2-1}-a\Big)^d+\Big(\sqrt{a^2-1}-a\Big)^d\Big).$$

To prove that the absolute point is an a'-fold point of the curve $CH(n,d,a)$, it is sufficient to prove that any line passing through the absolute point intersects the curve at $2(n+d)$ points, a' of which coincide with that point. If we switch to the homogeneous coordinates ($x=\frac{x_1}{x_0}$, $y=\frac{x_2}{x_0}$), the above equations turn into:

– if n is an even number (d must be odd)

$$(x_1^2+x_2^2)^{n+1}\Big(\sum_{j=0}^{\lfloor\frac{d}{2}\rfloor}\sum_{k=0}^{j}\sum_{l=0}^{\lfloor\frac{d-2k}{2}\rfloor}(-1)^{d-k-1}\binom{d}{2j}\binom{j}{k}\binom{d-2k}{2l+1}a^{d-2k-2l-1}(x_1^2+x_2^2)^l x_0^{d-2l-1}\Big)^2$$

(2.14)

$$=\Big(\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^i\binom{n}{2i}x_1^{n-2i}x_2^{2i}x_0^d - (x_1^2+x_2^2)^{\frac{n}{2}}\sum_{j=0}^{\lfloor\frac{d}{2}\rfloor}\sum_{k=0}^{j}\sum_{l=0}^{\lfloor\frac{d-2k}{2}\rfloor}(-1)^{d-k}\binom{d}{2j}\binom{j}{k}\binom{d-2k}{2l}a^{d-2k-2l}(x_1^2+x_2^2)^l x_0^{d-2l}\Big)^2$$

– if n is an odd number

$$(x_1^2+x_2^2)^{n}\Big(\sum_{j=0}^{\lfloor\frac{d}{2}\rfloor}\sum_{k=0}^{j}\sum_{l=0}^{\lfloor\frac{d-2k}{2}\rfloor}(-1)^{d-k}\binom{d}{2j}\binom{j}{k}\binom{d-2k}{2l}a^{d-2k-2l}(x_1^2+x_2^2)^l x_0^{d-2l}\Big)^2$$

(2.15)

$$=\Big(\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^i\binom{n}{2i}x_1^{n-2i}x_2^{2i}x_0^d - (x_1^2+x_2^2)^{\frac{n+1}{2}}\sum_{j=0}^{\lfloor\frac{d}{2}\rfloor}\sum_{k=0}^{j}\sum_{l=0}^{D}(-1)^{d-k-1}\binom{d}{2j}\binom{j}{k}\binom{d-2k}{2l+1}a^{d-2k-2l-1}(x_1^2+x_2^2)^l x_0^{d-2l-1}\Big)^2$$

Every straight line t passing through the absolute point $F_1(0,1,i)$ has the equation of the form

$$x_2 = ix_1 + mx_0. \tag{2.16}$$

It intersects the curve $CH(n,d,a)$ at the points whose coordinates satisfy the equation obtained when x_2 from (2.16) is substituted in (2.14) and (2.15). Some short calculations deliver the following conclusions: if $n>d$, for every line t, $x_0=0$ is the solution of the obtained equations with multiplicity $a'=2d$, and $F_1(0,1,i)$ is the intersection of t and $CH(n,d,a)$ with the intersection multiplicity $a'=2d$. If $d>n$, for every line t, $x_0=0$ is the solution of the obtained equations with multiplicity $a'=n+d$, and $F_1(0,1,i)$ is the intersection of t and $CH(n,d,a)$ with the intersection multiplicity $a'=n+d$. To summarize: the absolute points are $2d$-fold points of $CH(n,d,a)$ when $d<n$, and they are $(n+d)$-fold points when $n<d$. □

3. Generalized Rose Surfaces

In paper [2] the authors considered a *congruence of circles* $\mathcal{C}(p)$ that consists of circles in Euclidean space $\mathbb{E}^3$ passing through two given points P_1, P_2. The points P_1, P_2 lie on the axis z and are given by the coordinates $(0, 0, \pm p)$, where $p = \sqrt{q}$, $q \in \mathbb{R}$. If q is greater, equal to or less than zero, i.e. the points P_1, P_2 are real and different, coinciding (the axis z is the tangent line of all circles of the congruence) or imaginary, the congruence $\mathcal{C}(p)$ is called an *elliptic*, *parabolic* or *hyperbolic* congruence, respectively.

For every point A ($A \notin z$), there exists a unique circle $c^A(p) \in \mathcal{C}(p)$ passing through the points A, P_1 and P_2. For every point at infinity A^∞ ($A^\infty \notin z$), the circle $c^{A^\infty}(p)$ splits into the axis z and the line at infinity in the plane ζ through the axis z and the point A^∞.

A point is defined as a singular point of a congruence if infinitely many curves pass through it. The singular points of $\mathcal{C}(p)$ are the points on the axis z and the absolute points of $\mathbb{E}^3$ [2].

For a given congruence $\mathcal{C}(p)$ and a given curve α, a *circular surface* $\mathcal{CS}(\alpha, p)$ is defined as the system of circles from $\mathcal{C}(p)$ that intersect α.

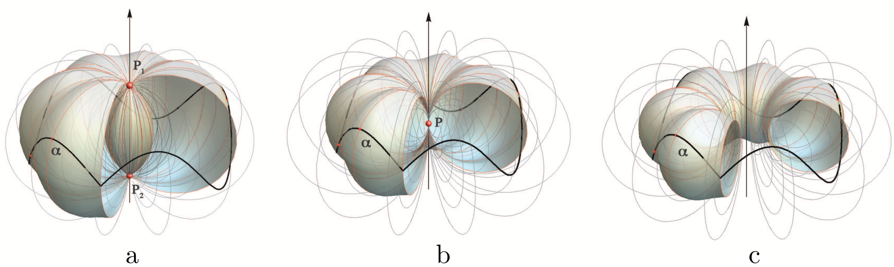


Figure 2: Some examples of $\mathcal{CS}(\alpha, p)$ when $\mathcal{C}(p)$ is an elliptic, parabolic and hyperbolic congruence are shown in figures a, b and c, respectively

If $\alpha\colon I \to \mathbb{R}^3$, $I \subseteq \mathbb{R}$, is given by $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$, $\alpha_i \in C^1(I)$, and if its normal projection on the plane $z = 0$ is denoted by $\alpha_{xy}(t) = (\alpha_1(t), \alpha_2(t), 0)$, then $\mathcal{CS}(\alpha, p)$ is given by the parametric equations:

$$x(t,\theta) = \frac{\alpha_1(t)}{2\left\|\alpha_{xy}(t)\right\|^2}\left(\sqrt{4p^2\left\|\alpha_{xy}(t)\right\|^2 + (\left\|\alpha(t)\right\|^2 - p^2)^2}\cos\theta + \left\|\alpha(t)\right\|^2 - p^2\right)$$

$$y(t,\theta) = \frac{\alpha_2(t)}{2\left\|\alpha_{xy}(t)\right\|^2}\left(\sqrt{4p^2\left\|\alpha_{xy}(t)\right\|^2 + (\left\|\alpha(t)\right\|^2 - p^2)^2}\cos\theta + \left\|\alpha(t)\right\|^2 - p^2\right)$$

(3.1)

$$z(t,\theta) = \sqrt{p^2 + \frac{(\left\|\alpha(t)\right\| - p^2)^2}{4\left\|\alpha_{xy}(t)\right\|^2}}\sin\theta, \qquad (t,\theta) \in I \times [0, 2\pi).$$

In [2] the authors showed that if α is an m^{th} order algebraic curve that cuts the axis z at z' points, the absolute conic at a' pairs of the absolute points and with the points P_1 and P_2 as p'_1-fold and p'_2-fold points, respectively, then, the following statements hold:

1. $\mathcal{CS}(\alpha, p)$ is an algebraic surface of the order $3m - (z' + 2a' + 2p'_1 + 2p'_2)$.
2. The absolute conic is an $m - (z' + p'_1 + p'_2)$-fold curve of $\mathcal{CS}(\alpha, p)$.
3. The axis z is an $(m - 2a' + z')$–fold line of $\mathcal{CS}(\alpha, p)$.
4. The points P_1, P_2 are $2m - (2a' + p'_1 + p'_2)$–fold points of $\mathcal{CS}(\alpha, p)$.

The singular lines and points of $\mathcal{CS}(\alpha, p)$ are the following:

5. The line z and the absolute conic are in the general case the singular lines of $\mathcal{CS}(\alpha, p)$. In particular, the points P_1, P_2 are of higher multiplicity than the other points on the line z.
6. If different points of the curve α determine the same circle $c \in \mathcal{C}(p)$, this circle is a singular circle of $\mathcal{CS}(\alpha, p)$ and its multiplicity equals the number of intersection points of α and c. Through every singular point of α, the singular circle of $\mathcal{CS}(\alpha, p)$ passes.
7. The intersection points of α and $c(0)$, where $c(0)$ is the circle given by the equations $x^2 + y^2 = -p^2$ $(p^2 < 0)$ and $z = 0$, are the singular points of $\mathcal{CS}(\alpha, p)$.

In this paper we study the properties of a special class of circular surfaces given by the following definition:

Definition 3.1. A generalized rose surface is a circular surface $\mathcal{CS}(\alpha, p)$ when the curve α is a generalized rose $CH(n, d, a)$ lying in a plane that does not contain the axis z.

Some generalized rose surfaces $\mathcal{CS}(\alpha, p)$ for $\alpha = CH(n, d, 0)$ and $p \in \mathbb{R}$ are simply called rose surfaces and were studied in detail in [1]. Their parametric and implicit equations were derived, their singularities were investigated and they were visualized by the program *Mathematica*. This paper presents the extension of [1] since we are also dealing with $\mathcal{CS}(\alpha, p)$ for $\alpha = CH(n, d, a)$ when $a \neq 0$ and $p \in \mathbb{C}$.

Let $CH(n, d, a)$ lie in any plane of $\mathbb{E}^3$ that does not contain the axis z, let its $2n$-fold (n-fold) point be denoted by O, and let its another real j-fold points be denoted by D^j.

According to the properties $1-4$ of $\mathcal{CS}(\alpha, p)$ and the properties of $CH(n, d, a)$ given in Table 1, a classification of generalized rose surfaces according to their order and the multiplicity of the axes z, the absolute points and the points P_1, P_2 can be made. Since this classification depends on the position of $CH(n, d, a)$ with respect to the singular points of the congruence $\mathcal{C}(p)$, the following cases should be taken into consideration:

Type 1: $O = P_1$ or $O = P_2$;

Type 2: $O \in z$, $O \neq P_i$, $i = 1, 2$;

Type 3: $D^j = P_1$ or $D^j = P_2$;

Type 4: $D^j \in z$, $D^j \neq P_i$, $i = 1, 2$;

Type 5: $\forall Z \in z$, $Z \notin CH(n, d, a)$.

The algebraic properties of these surfaces are given in the following theorem:

Theorem 3.2. *For generalized rose surfaces $\mathcal{CS}(CH(n, d, a), p)$, the following table is valid:*

Type of $\mathcal{CS}(CH, p)$			order of $\mathcal{CS}(CH, p)$	multiplicity of the absolute conic	multiplicity of the axis z	multiplicity of the points P_i
1A	$a = 0$	$d < n$	$n + d$	d	$n - d$	n
	$n \cdot d$ odd	$d > n$	$2d$	d	0	d
1B	other	$d < n$	$2(n + d)$	$2d$	$2(n - d)$	$2n$
	cases	$d > n$	$4d$	$2d$	0	$2d$
2A	$a = 0$	$d < n$	$2n + d$	d	$2n - d$	$2n$
	$n \cdot d$ odd	$d > n$	$n + 2d$	d	n	$n + d$
2B	other	$d < n$	$2(2n + d)$	$2d$	$2(2n - d)$	$4n$
	cases	$d > n$	$2(n + 2d)$	$2d$	$2n$	$2(n + d)$
3A	$a = 0$	$d < n$	$3n + d - 2j$	$n + d - j$	$n - d$	$2n - j$
	$n \cdot d$ odd	$d > n$	$2(n + d) - 2j$	$n + d - j$	0	$n + d - j$
3B	other	$d < n$	$2(3n + d) - 2j$	$2(n + d) - j$	$2(n - d)$	$4n - j$
	cases	$d > n$	$4(n + d) - 2j$	$2(n + d) - j$	0	$2(n + d) - j$
4A	$a = 0$	$d < n$	$3n + d - j$	$n + d - j$	$n - d + j$	$2n$
	$n \cdot d$ odd	$d > n$	$2(n + d) - j$	$n + d - j$	j	$n + d$
4B	other	$d < n$	$2(3n + d) - j$	$2(n + d) - j$	$2(n - d) + j$	$4n$
	cases	$d > n$	$4(n + d) - j$	$2(n + d) - j$	j	$2(n + d)$
5A	$a = 0$	$d < n$	$3n + d$	$n + d$	$n - d$	$2n$
	$n \cdot d$ odd	$d > n$	$2(n + d)$	$n + d$	0	$n + d$
5B	other	$d < n$	$2(3n + d)$	$2(n + d)$	$2(n - d)$	$4n$
	cases	$d > n$	$4(n + d)$	$2(n + d)$	0	$2(n + d)$

Table 2: Properties of $\mathcal{CS}(\alpha, p)$

Proof. The statements regarding generalized rose surfaces are the immediate consequence of the algebraic properties 1 - 4 of the circular surfaces $\mathcal{CS}(\alpha, p)$ when α is the cyclic harmonic curve $CH(n, d, a)$ with the properties given in Table 1. More precisely, the values m and a' are given in the second and fourth columns of Table 1, while the values p'_1, p'_2 and z' depend on the position of $CH(n, d, a)$ with respect to the line z as follows:
Type 1: Let us assume that $O = P_1$. Then $z' = p'_2 = 0$, while $p'_1 = n$ when $a = 0$ and $n \cdot d$ is odd, and $p'_1 = 2n$ in other cases.
Type 2: $p'_1 = p'_2 = 0$, $z' = n$ when $a = 0$ and $n \cdot d$ is odd, and $z' = 2n$ in other cases.
Type 3: If we assume that $D^j = P_1$, then $p'_1 = j$, $p'_2 = z' = 0$.
Type 4: $p'_1 = p'_2 = 0$, $z' = j$.
Type 5: $p'_1 = p'_2 = z' = 0$.

By substituting m, a', p'_1, p'_2 and z' into expression given in statements 1-4, the algebraic properties of $\mathcal{CS}(CH(n, d, a), p)$ are obtained. □

4. Vizualization of Generalized Rose Surfaces

In this section we give some examples of generalized rose surfaces. For a particular surface, parametric and implicit equations can be obtained on the basis of the equation (3.1). We check the properties given in Theorem 3.2 and visualize the shape of the surface. For computing and plotting we use the program *Mathematica*.

The circular surfaces $\mathcal{CS}(CH, 0)$ for $CH = CH(7, 3, a)$, where $a = 0, 0.25, 1$ and 2.5, are shown in Figures 3 a, b, c and d, respectively. The curve CH lies in the plane $z = 0$ and the point O coincides with the directing point $P_1 = P_2$ of the parabolic congruences $\mathcal{C}(0)$. Therefore, CH in the case a is an algebraic curve of the order 10 having the absolute conic as a triple curve, axis z as a quadruple line and point O as a 14-fold point. The other three surfaces are of the order 20 and the absolute conic is a 6-fold curve, the axis z is an 8-fold line and the point O is a 14-fold point of the surfaces.

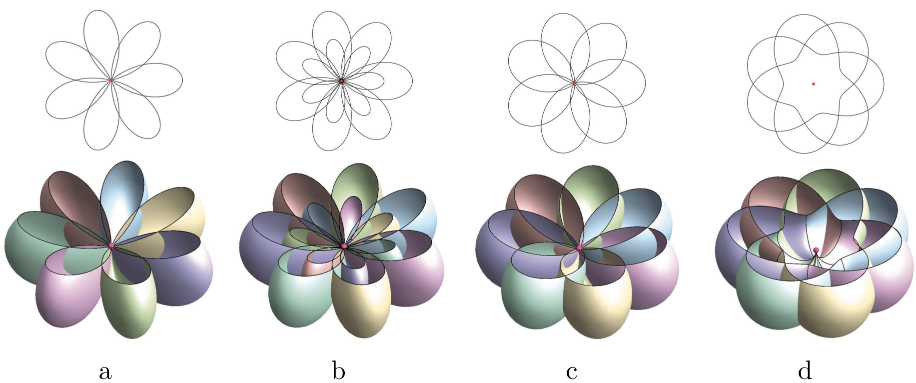


Figure 3: Four surfaces $\mathcal{CS}(CH, 0)$ of type 1

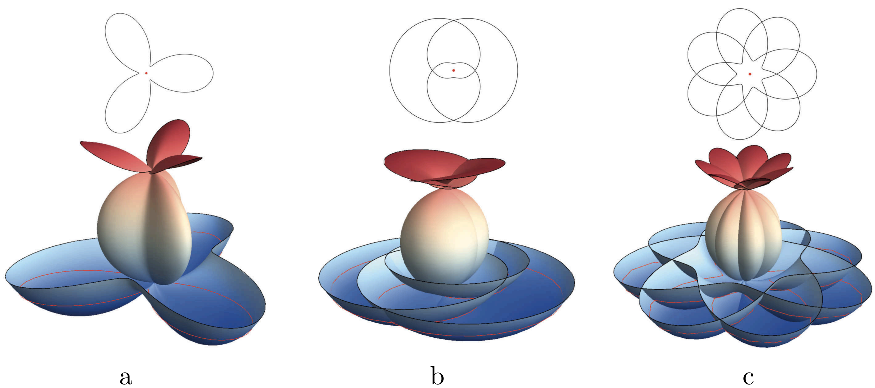
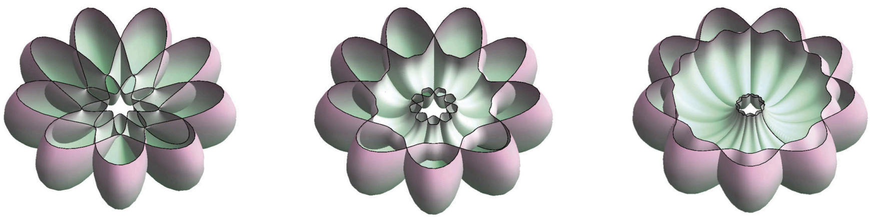


Figure 4: Three surfaces $\mathcal{CS}(CH, 1)$ of type 1B

In Figure 4 three generalized rose surfaces $\mathcal{CS}(CH,1)$ of type 1B are presented. CH in Figures 4 a, b and c are curtates $CH(3,1,1.25)$, $CH(2,3,1.25)$ and $CH(7,3,1.25)$ in the plane $z=-1$ with the $2n$-fold isolated point O coinciding with the directing point P_1 of the elliptic congruence $\mathcal{C}(1)$, respectively.

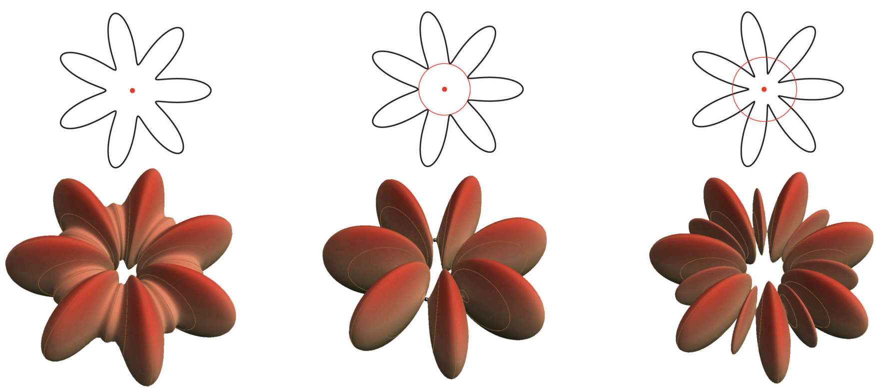
If a circular surface is defined by a hyperbolic congruence and a curtate with the $2n$-fold point on the axis z, the surface is of type 2B. The generalized rose surfaces $\mathcal{CS}(CH,i)$ for $CH(9,2,2)$ in the plane $z=0,0.5,1$ are shown in Figures 5 a, b, c, respectively. All three surfaces $\mathcal{CS}(CH,i)$ are of the order 40. The absolute conic is a quadruple curve and the axis z is a 32-fold line of the surfaces.



a b c

Figure 5: Three surfaces $\mathcal{CS}(CH,i)$ of type 2B

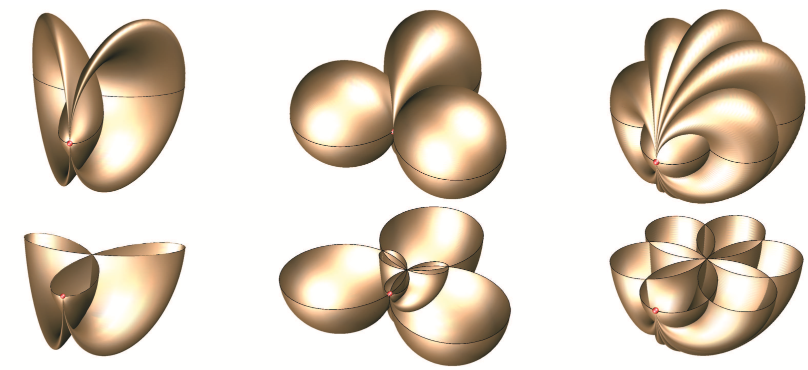
The surfaces $\mathcal{CS}(CH,i)$ for $CH(7,1,2)$ in the plane $z=0.75,0$ are shown in Figures 6 a and b, respectively. The surface in Figure 6 c is defined by $CH(7,1,1.5)$ in the plane $z=0$. All three surfaces are of type 2B and of the order 30, with the absolute conic as a double curve, axis z as a 26-fold line and the points P_1,P_2 as the 28-fold points. The surfaces in Figures 6 b and c possess the singular points in the intersection points of CH and $c(0)$, where $c(0)$ is the circle given by the equations $x^2+y^2=1$, $z=0$.



a b c

Figure 6: Three surfaces $\mathcal{CS}(CH,i)$ of type 2B

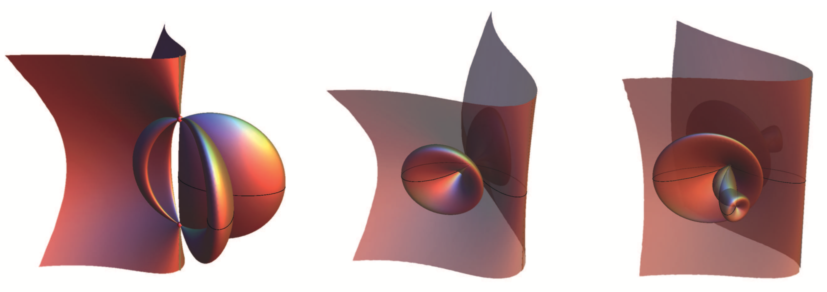
The surface $\mathcal{CS}(CH, 0)$ shown in Figure 7 a is defined by the parabolic congruence $\mathcal{C}(0)$ and the rose $CH(3, 1, 0)$ intersecting the axis z at its regular point $D^1 = P_1 = P_2$. The obtained surface is of type 3A and the order 8. The absolute conic is a triple conic, the axis z is a double line and D^1 is an eightfold point of the surface. The surfaces $\mathcal{CS}(CH, 0)$ in Figures 7 b and c are of type 3B. The $CH(3, 2, 0.5)$ in Figure 7 b has a triple point at the directing point of the congruence $\mathcal{C}(0)$, while $CH(3, 2, 0)$ in Figure 7 c has a regular point at that point.



a b c

Figure 7: Three surfaces $\mathcal{CS}(CH, 0)$ of type 3

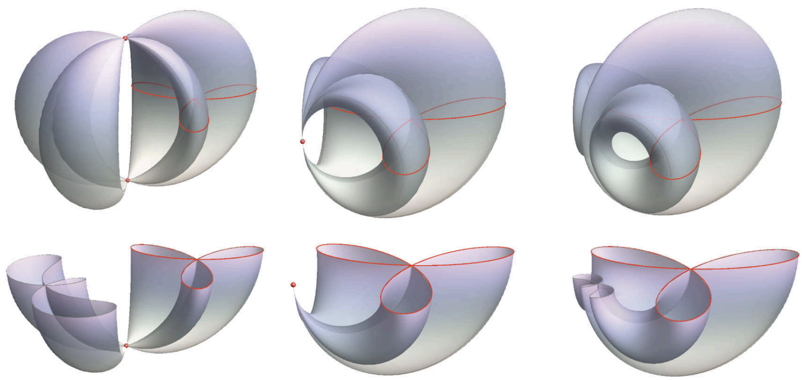
All surfaces of type 4 pass through the real lines at infinity. Three such surfaces are presented in Figure 8. $\mathcal{CS}(CH, p)$ in case a is defined by $CH(3, 1, 0)$, and $p = 1$, $\mathcal{CS}(CH, p)$ in case b is defined by $CH(3, 1, 0)$ and $p = i$, and $\mathcal{CS}(CH, p)$ in case c is defined by $CH(5, 1, 0)$ and $p = i$. All three roses lie in the plane $z = 0$ and intersect the axis z at the regular point D^1. The first two surfaces are of the order 9 with the absolute conic as a triple curve, axis z as a triple line and the directing points as the sixfold points. The third surface is of the order 15 with the absolute conic as a 5-fold curve, axis z as a 5-fold line and the directing points as the 10-fold points.



a b c

Figure 8: Three surfaces $\mathcal{CS}(CH, p)$ of type 4A

Three surfaces of type 5A defined by $CH(3,1,0)$ and $p = 1, 0, i$ are shown in Figure 9 a, b, c, respectively. $CH(3,1,0)$ lies in the plane $z = 0$ and has a triple point at the point $(1,0,0)$. All three surfaces are of the order 10, have the absolute conic as a quadruple curve, axis z as a double line and the directing points as the sixfold points.



a b c

Figure 9: Three surfaces $\mathcal{CS}(CH, p)$ of type 5A

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