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PEDAL SURFACES OF FIRST ORDER LINE CONGRUENCES

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Abstract.

This paper is a short overview of the deducing of the pedal surfaces $\mathcal{P}_n^{n+2}$ for the first order line congruences $\mathcal{C}_n^1$. $\mathcal{P}_n^{n+2}$ pass through the absolute conic of Euclidean space and are $(n+2)$-order surfaces with n-ple straight line. We described their construction and derived their parametric equations. These equations enable Mathematica visualizations of $\mathcal{P}_n^{n+2}$ and they are given in two examples ($\mathcal{P}_4^6$ and $\mathcal{P}_{2k}^{2k+2}$).

Keywords

congruence of lines, pedal surface of congruence

1 INTRODUCTION

A congruence $\mathcal{C}$ is a double infinite line system, i.e. it is the set of lines in a three-dimensional space (projective, affine or Euclidean) depending on two parameters. A line $l \in \mathcal{C}$ is said to be a *ray* of the congruence. The *order* of a congruence is the number of its rays which pass through an arbitrary point; the *class* of a congruence is the number of its rays which lie in an arbitrary plane. *m*th *order*, *n*th *class* congruence is denoted $\mathcal{C}_n^m$. A point is the *singular point* of a congruence if ∞^1 rays (1-parametrically infinite lines) pass through it. A plane is the *singular plane* of a congruence if it contains ∞^1 rays.

In Euclidean space $\mathbb{E}^3$, the *pedal surface* of a congruence $\mathcal{C}$ with respect to a pole P is the locus of the feet of perpendiculars from a point P to the rays of a congruence $\mathcal{C}$. If $\mathcal{C}$ is mth order nth class congruence, the order of its pedal surface for the pole P is $2m + n$, [6].

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2 PEDAL SURFACES OF FIRST ORDER LINE CONGRUENCES

2.1 Directing lines of $\mathcal{C}_n^1$

According to [7, p. 64], [10, pp. 1184-1185], there are only two types of the first order congruences. The first one are nth class congruences and their rays are transversals of one straight line d and one nth order space curve c^n which cuts this straight line in $n-1$ points (see Fig. 1a). The intersection points of d and c^n can be the multiple points of c^n (with the highest multiplicity $n-2$) or some of them can coincide (there are cases when d is the tangent line of c^n, the tangent at inflection, etc.). The second type are only 3rd class congruences and its rays cut a twisted cubic twice. The properties of the first order congruences (the construction of its rays, singular points and planes, focal properties, etc.) can be found in [1].

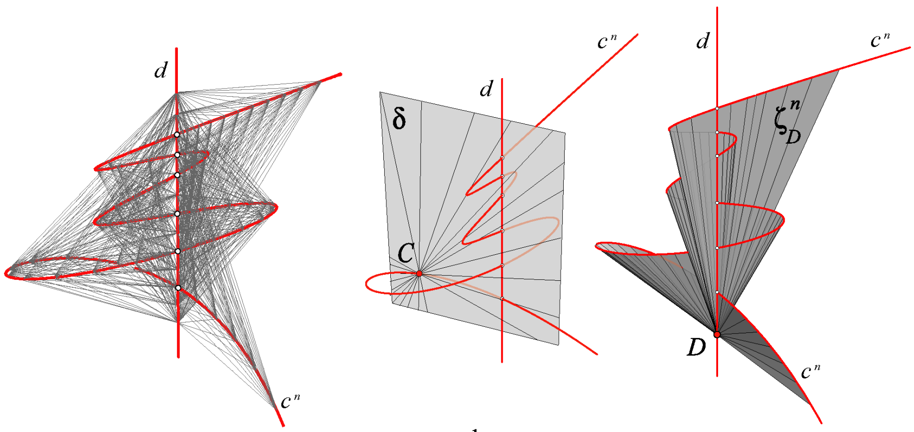


Figure 1: The rays of $\mathcal{C}_n^1$ are transversals of d and c^n (a). The singular points of C_n^1 lie on its directing lines: for $C \in c^n$ they form a pencil of lines in the plane through d (b) and for $D \in d$ they form an nth degree cone with the vertex D (c). The singular planes of C_n^1 are the planes δ through d.

2.2 Construction of pedal surface $\mathcal{P}_n^{n+2}$

In [2] the authors defined one transformation of three-dimensional projective space where corresponding points lie on the rays of the 1st order, nth class congruence $\mathcal{C}_n^1$ and are conjugate with respect to some proper quadric Ψ. This transformation, called $(n+2)$-degree inversion, maps a straight line to an $(n+2)$-order space curve and a plane to an $(n+2)$-order surface which contains n-ple straight line. According to [2], the pedal surfaces of the first type congruence $\mathcal{C}_n^1$ with respect to a pole P is the image of the plane at infinity given by the $(n+2)$-degree inversion with respect to $\mathcal{C}_n^1$ and any sphere with the center P. Thus, it is an $(n+2)$-order surface with n-ple straight line d and contains the absolute conic. It will be denoted $\mathcal{P}_n^{n+2}$.

It is clear that any plane through the n-ple line of an $(n+2)$-order surface cuts this surface in its n-ple line and one conic. If the surface contains the absolute conic, this conic is a circle.

In any plane δ trough the directing straight line d the rays of $\mathcal{C}_n^1$ form the pencil of lines (C), where a point $C \notin d$ is the intersection of the plane δ and the directing curve c^n [1], see Fig. 2a. If a pole P is in the general position to the directing lines of a congruence $\mathcal{C}_n^1$, the feet of perpendiculars from P to the rays of the pencil (C) form the circle c with the diameter $\overline{CP'}$, where P' is the orthogonal projection of P to δ, see Fig. 2b. The proof of this statement is elementary.

For given pole P, the path of the point P' is the circle k which lies in the plane through P perpendicular to d. The diameter of k is $\overline{PP_d}$, where P_d is the normal projection of P to d, see Fig. 2c. Thus, we can regard the pedal surface $\mathcal{P}_n^{n+2}$ as the system of circles in the planes through the n-ple line d with the end points of diameters on the curve c^n and the circle k.

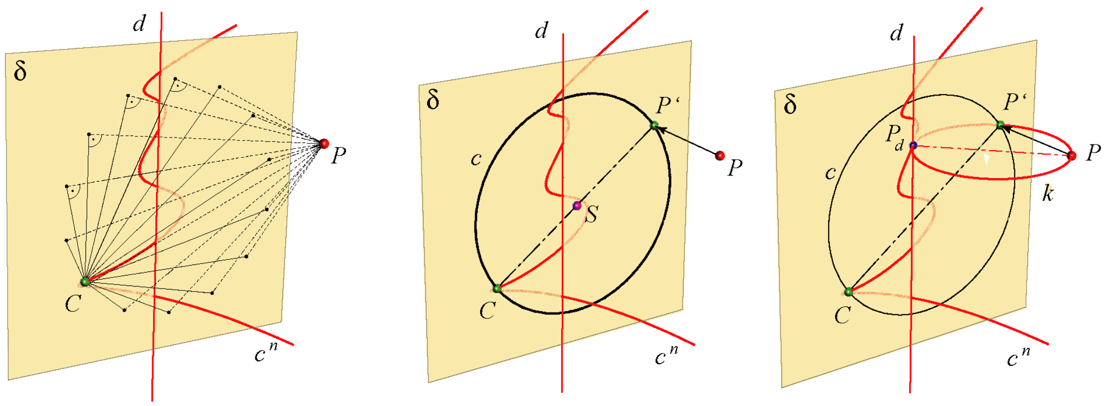


Figure 2: One system of curves on $\mathcal{P}_n^{n+2}$ can be construct as the circles in the planes through d with the end points of diameters on c^n and k.

2.3 Parametric equations of $\mathcal{P}_n^{n+2}$

Let the directing straight line of $\mathcal{C}_n^1$ be the axis z, and let the directing curve c^n be given by the following parametrization:

$$\mathbf{r}_{c^n}(\varphi) = (x_{c^n}(\varphi), y_{c^n}(\varphi), z_{c^n}(\varphi)), \quad x_{c^n}, y_{c^n}, z_{c^n} : [0, \pi) \to \mathbb{R}. \tag{1}$$

Let (p_x, p_y, p_z) be the coordinates of the pole P.

Let (r, z), where $|r| = \sqrt{x^2 + y^2}$, be the coordinates of the points in the plane $\delta(\varphi)$, which is given by equation $y = x \tan\varphi$ if $\varphi \in [0, \pi)$, $\varphi \neq \pi/2$, and $x = 0$ if $\varphi = \pi/2$, see Fig. 3.

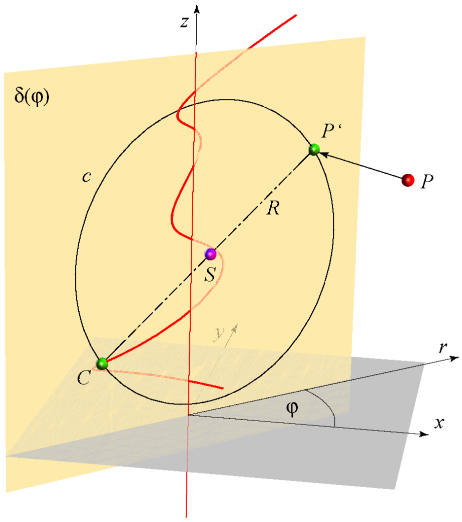


Figure 3: (r, z) are Cartesian coordinates in the plane $\delta(\varphi)$.

The coordinates of points $C, P' \in \delta(\varphi)$ are

$$\begin{aligned} r_C(\varphi) &= \sqrt{x_{c^n}(\varphi)^2 + y_{c^n}(\varphi)^2} \\ z_C(\varphi) &= z_{c^n}(\varphi) \\ r_{P'}(\varphi) &= p_x \cos\varphi + p_y \sin\varphi \\ z_{P'}(\varphi) &= p_z. \end{aligned} \tag{2}$$

$R(\varphi)$ is the radius and $S(r_S(\varphi), z_S(\varphi))$ is the center of the circle c in the plane $\delta(\varphi)$.

$$\begin{aligned} R(\varphi) &= \frac{\sqrt{(r_C(\varphi) - r_{P'}(\varphi))^2 + (z_C(\varphi) - p_z)^2}}{2} \\ r_S(\varphi) &= \frac{r_C(\varphi) + r_{P'}(\varphi)}{2} \\ z_S(\varphi) &= \frac{z_C(\varphi) + p_z}{2} \end{aligned} \tag{3}$$

Since the parametric equations of the circle c in the plane $\delta(u)$ are

$$\begin{aligned} r(\theta) &= R(\varphi)\sin\theta + r_S(\varphi) \\ z(\theta) &= R(\varphi)\cos\theta + z_S(\varphi), \quad \theta \in [0, 2\pi), \end{aligned} \tag{4}$$

therefore the parametric equations of the surface $\mathcal{P}_n^{n+2}$ are the following

$$\begin{aligned} x(\theta,\varphi) &= \cos\varphi\,(R(\varphi)\sin\theta + r_S(\varphi)) \\ y(\theta,\varphi) &= \sin\varphi\,(R(\varphi)\sin\theta + r_S(\varphi)) \\ z(\theta,\varphi) &= R(\varphi)\cos\theta + z_S(\varphi), \qquad \varphi \in [0,\pi),\ \theta \in [0, 2\pi). \end{aligned} \tag{5}$$

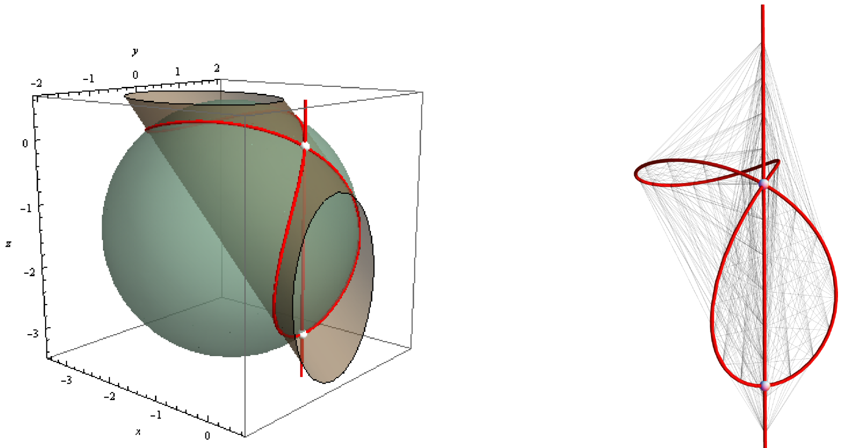
3 SPECIAL SEXTICS WITH QUADRUPLE LINE

Let the directing lines of a congruence $\mathcal{C}$ be the axis z and Viviani's curve (see Fig. 4a) which is the intersection of the following sphere and cylinder:

$$(x+\sqrt{2})^2 + y^2 + (z+\sqrt{2})^2 = 4, \quad (x+z+\sqrt{2})^2 + 2y^2 = 2. \tag{6}$$

From equations (6), by using the substitution $y \to x \tan u$, we obtain the following parametrization of Viviani's curve:

$$\mathbf{r}(\varphi) = 4\sqrt{2}\,\frac{1+3\cos 2\varphi}{(3+\cos 2\varphi)^2}\Big(-2(\cos\varphi)^2, -\sin 2\varphi, (\sin\varphi)^2\Big), \qquad \varphi \in [0,\pi). \tag{7}$$



a b

Figure 4: The rays of $\mathcal{C}_4^1$ are transversals of the axis z and Viviani's curve given by eq. (7).

The axis z cuts Viviani's curve in two points, $S_1 = (0,0,0)$ and $S_2 = (0,0,-2\sqrt{2})$, where S_1 is the double point of Viviani's curve. Since Viviani's curve is the 4th order space curve c^4 and the axis z cuts it in 3 points, then the transversals of z and c^4 form the 1st order and 4th class congruence. The directing lines and some rays of $\mathcal{C}_4^1$ are shown in Fig. 4b.

According to [2], the pedal surfaces of this $\mathcal{C}_4^1$ are 6th order surfaces (sextics) with a quadruple line through the axis z. In this case the coordinates r_C and z_C from eq. 2 are:

$$\begin{aligned} r_C(\varphi) &= = -8\sqrt{2}\,\frac{(1+3\cos 2\varphi)\cos\varphi}{(3+\cos 2\varphi)^2} \\ z_C(\varphi) &= 4\sqrt{2}\,\frac{(1+3\cos 2\varphi)\sin^2\varphi}{(3+\cos 2\varphi)^2}. \end{aligned} \tag{8}$$

From these equations, eq. (3) and eq. (5) we obtain the parametric equations of $\mathcal{P}_4^6$ which depend only on the coordinates of a pole P and enable *Mathematica* visualizations of $\mathcal{P}_4^6$. Some examples are given in Fig. 5 and Fig. 6.

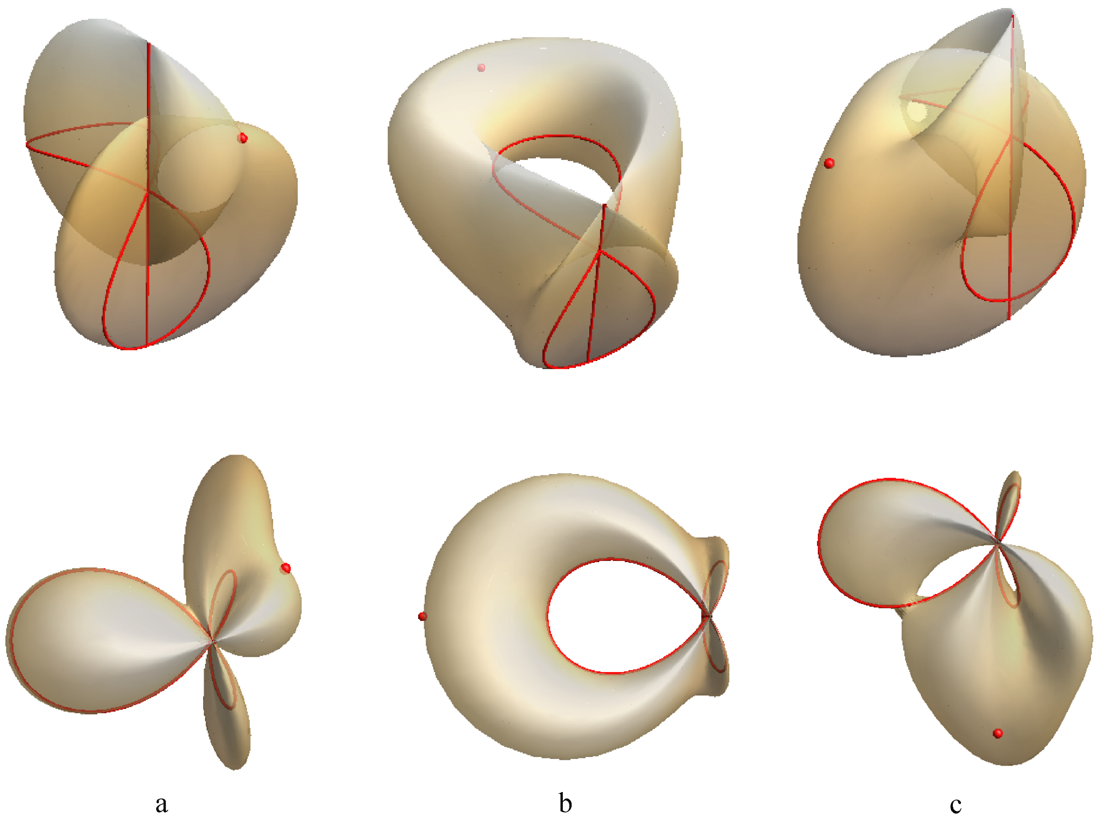
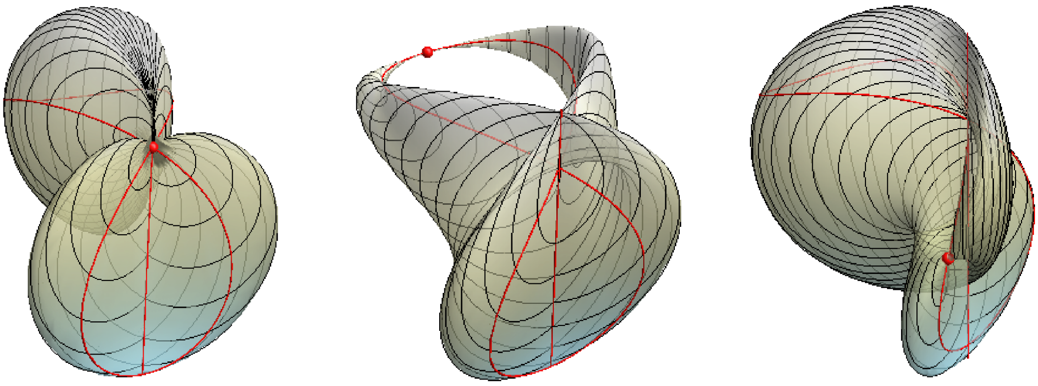


Figure 5: Three pedal surfaces of $\mathcal{C}_4^1$ with respect to the poles $(1,1,1)$, $(-5,0,0)$ and $(0,-3,0)$ are shown in figure a, b and c, respectively. The directing lines of $\mathcal{C}_4^1$ and the poles are pointed out. Each surface is viewed from two different viewpoints.

In [3] we derived the implicit equation of $\mathcal{P}_4^6$ and studied the properties of their singularities. The following propositions are proved:

- The surface $\mathcal{P}_4^6$ has a quintuple point on the axis z iff the pole P lies on the axis z. In this case it is the unique quintuple point of $\mathcal{P}_4^6$. For different positions of P, we obtained five types of the fifth degree tangent cone at quintuple point.
- The surface $\mathcal{P}_4^6$ has twelve pinch-points on the quadruple line z (real or complex). There are six types of such points.
- The surface $\mathcal{P}_4^6$ has at least one real double point out of z iff the pole P lies on one 5th degree ruled surface. It has exactly two real double points out of z iff the pole P lies on the part of one parabola.



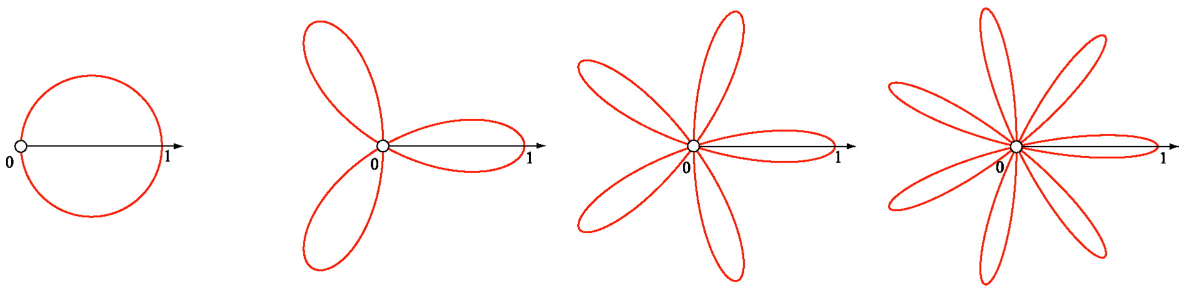
a b c

Figure 6: If the pole P lies on axis z, all circles c pass through it and P is the quintuple point of $\mathcal{P}_4^6$ (case a for $P(0,0,0)$. If the pole P lies on Viviani's curve, the circle c through it splits into the isotropic lines in the plane δ trough P and P is the double point of $\mathcal{P}_4^6$ (case b for $\mathbf{r}_P(0°)$ and case c for $\mathbf{r}_P(110°)$).

4 PEDAL SURFACES $\mathcal{P}_{2k}^{2k+2}$

A special class of $\mathcal{C}_n^1$ arises if all intersection points of the directing lines d and c^n coincide. In this case c^n is a plane curve with one singular point of the highest multiplicity $n-1$, and a line d passes through this point. Here we will regard a special $\mathcal{C}_n^1$ where n is an even number and a directing curve c^{2k} is a $(2k-1)$-folium given by the following polar equation:

$$r(\varphi) = \cos(2k-1)\varphi, \quad \varphi \in [0, \pi). \tag{9}$$



a b c d

Figure 7: $(2k-1)$-foliums for $k = 1, 2, 3, 4$ are shown in figures a, b, c and d, respectively.

According to the multiple-angle formula, $\cos(2k-1)\varphi$ can be displayed as

$$\sum_{i=0}^{k}(-1)^i C_{2i}^{2k-1}(\cos\varphi)^{2k-1-2i}(\sin\varphi)^{2i} \tag{10}$$

where C_{2i}^{2k-1} is a binomial coefficient. Therefore, from eq. (9), by using the substitutions $r(\varphi) = \sqrt{x^2+y^2}$, $\cos\varphi = \frac{x}{\sqrt{x^2+y^2}}$ and $\sin\varphi = \frac{y}{\sqrt{x^2+y^2}}$, we obtain the following implicit equation of $(2k-1)$-folium:

$$(x^2+y^2)^k - \tau^{2k-1} = 0, \quad \text{where } \tau^{2k-1} = \sum_{i=0}^{k}(-1)^i C_{2i}^{2k-1} x^{2k-1-2i} y^{2i}. \tag{11}$$

From eq. (11) it is clear that $(2k-1)$-folium is $2k$-order curve c^{2k}, with $(2k-1)$-ple point at the origin, where $2k-1$ tangent lines at it are given by equation $\tau^{2k-1} = 0$, [8, p. 27].

Let the axis z and $(2k-1)$-folium c^{2k} in the plane $z = 0$ be the directing lines of the congruence $\mathcal{C}_{2k}^1$ (see Fig. 8a). The pedal surfaces of this congruence is a $(2k+2)$-order surface with $2k$-ple axis z.

If $(p_x, p_y, p_z) \in \mathbb{R}^3$ are the coordinates of a pole P, then the diameters $\overline{CP'}$ of the circles $c \subset \delta(\varphi)$ (see Fig. 8b) are determined by the following coordinates:

$$C = (\cos(2k-1)\varphi, 0), \quad P' = (p_x\cos\varphi + p_y\sin\varphi, p_z). \tag{12}$$

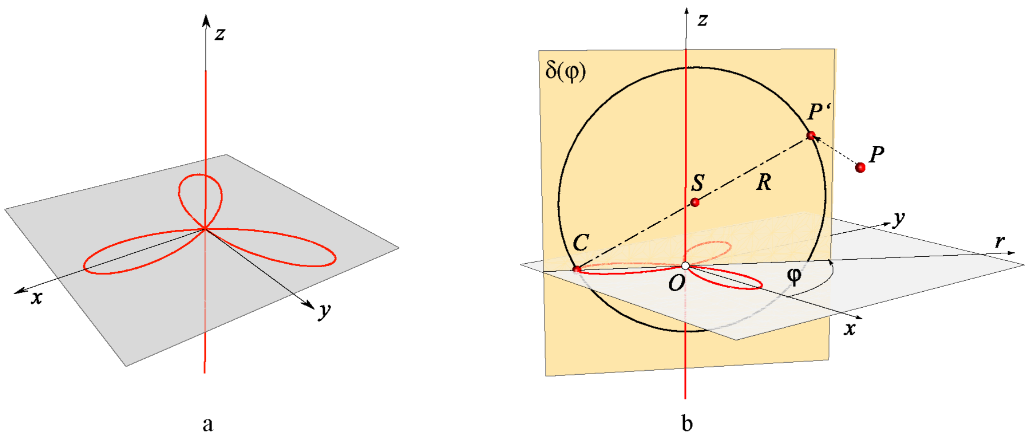
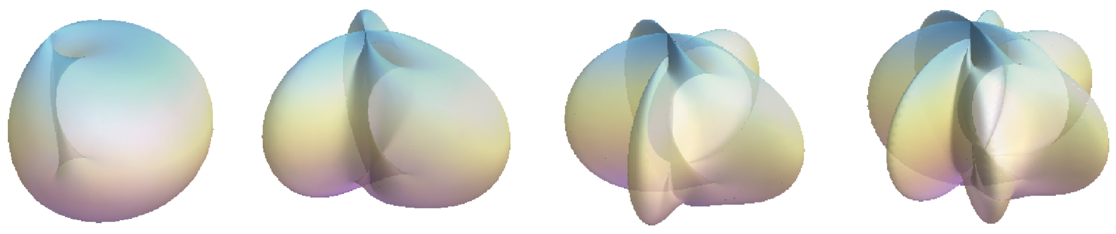


Figure 8: The directing lines of $\mathcal{C}_{2k}^1$ for $k = 2$ (a), and the circle $c \subset \delta$ which lies on the pedal surface of $\mathcal{C}_{2k}^1$.

Now, from eq. (3) and eq. (5) we obtain the parametric equations of $\mathcal{P}_{2k}^{2k+2}$ which depend on the coordinates of a pole P and the number k which determines the folium c^{2k}. They enable *Mathematica* visualizations of $\mathcal{P}_{2k}^{2k+2}$ which are shown in Fig. 9.

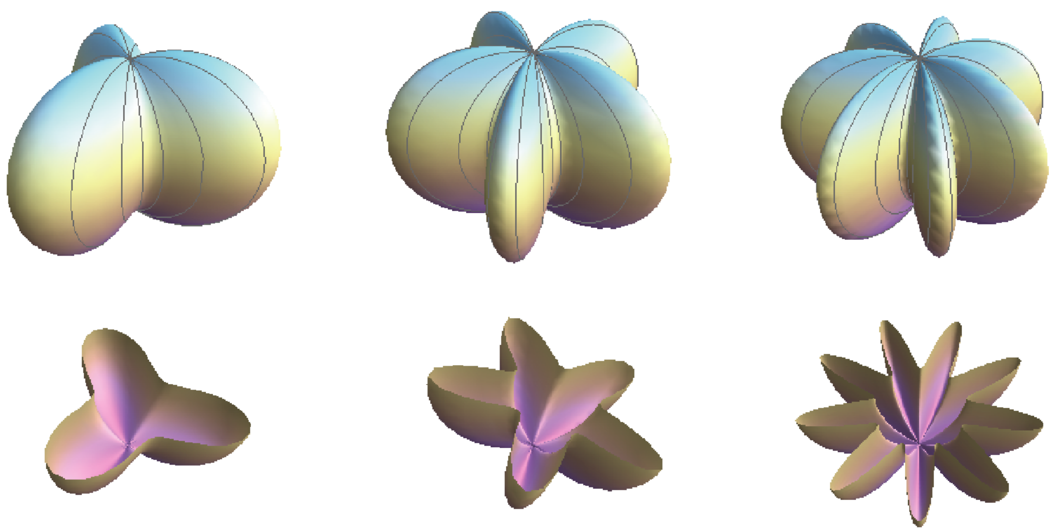


a b c d

Figure 9: $\mathcal{P}_{2k}^{2k+2}$ for $P(1,0,2)$ and $k = 1,2,3,4$ are shown in figures a, b, c and d, respectively.

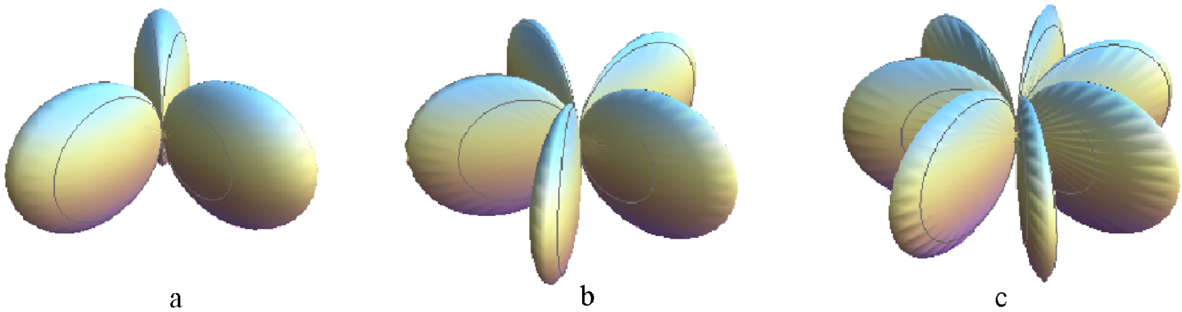
These surfaces are elaborated in detail in [4]. Here we point out only one interesting property: If the pole P lies on the axis z, $\mathcal{P}_{2k}^{2k+2}$ splits into the pair of isotropic planes through z and a $2k$-order surface which is given by the following equation

$$P^{2k}(x,y,z) = (x^2+y^2)^{k-1}(x^2+y^2+z^2-p_z z) - \tau^{2k-1} = 0. \tag{13}$$



a b c

Figure 10: $\mathcal{P}_{2k-2}^{2k}$ for $P(0,0,2)$ given by equation $P^{2k}(x,y,z) = 0$ for $k = 2,3,4$ are shown in figures a, b and c, respectively.



a b c

Figure 11: $\mathcal{P}^{2k}_{2k-2}$ for $P(0,0,0)$ given by equation $P^{2k}(x,y,z) = 0$ for $k = 2,3,4$ are shown in figures a, b and c, respectively.

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