Visualisations of Gussian and Mean Curvatures by Using *Mathematica* and *webMathematica*

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Abstract. In this paper we have given a short overview on calculation of the Gaussian and mean curvatures of a regular surface and in six examples we have shown visualisations of the properties of that functions by using *Mathematica* and colour function `Hue`. We have described the program *webMathematica* and presented one web page which is powered by this program.

1. Introduction

Specific conditions concerning the installation of *Mathematica* in Croatia (the program is available on all university computers) stimulated some teachers of geometry and mathematics at the Faculties of Civil Engineering and Geodesy to try to improve teaching and learning process by means of *Mathematica* and *webMathematica*. Within the IT project *Selected Chapters of Geometry and Mathematics Treated by Means of Mathematica for Future Structural Engineers*[1] we designed educational material which enhances visually standard lectures and stimulates interactive and tutorial way of learning on the Internet. The parts of the educational material that we have created so far can be found, mostly in Croatian language, at the following address: *http://www.grad.hr/itproject_math/*

In this paper we present the part of that educational material related to the Gaussian and mean curvatures of a regular surface, which has been translated into English.

2. *Mathematica* visualisations of Gaussian and mean curvatures

For future structural engineers it is important to have the knowledge of the Gaussian and mean curvatures. For example: Tensile fabric structure (e.g. membrane roof) in a uniform state of tensile prestress behaves like a soap film stretched over a wire which is bent in a shape of a closed space curve. Soap film assumes a form which has the minimal area relative to all other surfaces stretched over the same wire; this surface is therefore called *minimal* surface. It can be shown that *mean curvature vanishes* at each point of that surface.

[1] The project has been supported by the Ministry of Science and Technology of the Republic of Croatia since 2002/03.

2.1. Gaussian and mean curvature of a regular surface

A *regular surface* $\Phi \subset \mathbb{R}^3$ is the set of points whose position vectors are the values of one-to-one continuous vector-valued function $\mathbf{r} : \mathcal{U} \to \mathbb{R}^3$

$$\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)) \tag{1}$$

where $\mathcal{U} \subset \mathbb{R}^2$ is open and connected, x, y, $z : \mathcal{U} \to \mathbb{R}$ are differentiable functions and

$$\forall (u, v) \in \mathcal{U}, \quad \mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v) \neq 0. \tag{2}$$

At each point of a regular surface a unique tangent plane and a normal vector exist. The unit normal vector $\mathbf{n}_0$ at the point with a position vector $\mathbf{r}(u, v)$ is given by the following formula:

$$\mathbf{n}_0 = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}. \tag{3}$$

The quadratic form $Edu^2 + 2Fdudv + Gdv^2$, where $E = \mathbf{r}_u \cdot \mathbf{r}_u$, $F = \mathbf{r}_u \cdot \mathbf{r}_v$ and $G = \mathbf{r}_v \cdot \mathbf{r}_v$, is called the *first fundamental form* of Φ.

The quadratic form $Ldu^2 + 2Mdudv + Ndv^2$, where $L = \mathbf{n}_0 \cdot \mathbf{r}_{uu}$, $M = \mathbf{n}_0 \cdot \mathbf{r}_{uv}$ and $N = \mathbf{n}_0 \cdot \mathbf{r}_{vv}$, is called the *second fundamental form* of Φ.

The *Gaussian curvature* K and the *mean curvature* H of a surface Φ are functions $K, H : \mathcal{U} \to \mathbb{R}$ given by the following formulas:

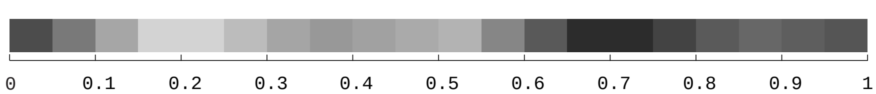
$$K = \frac{LN - M^2}{EG - F^2}, \quad H = \frac{EN - 2FM + GL}{2(EG - F^2)}. \tag{4}$$

2.2. Visualisations by using *Mathematica*

The geometrical interpretations of normal, Gaussian and mean curvatures of a regular surface can be found at the following address:
http://www.grad.hr/itproject_math/Links/sonja/gausseng/gausseng.html

According to Eq. 4 we can define (in the program *Mathematica*) the functions `gcurvature` and `mcurvature` [3, p.394] which calculate the Gaussian and mean curvatures at each point of a regular surface. These functions enable us to plot the graphs of the Gaussian and mean curvatures of regular surfaces and to colour surfaces with colours which depend of that curvatures.

For the following visualizations we used the periodical *Mathematica* colour function `Hue` (period 1). In black-white print this colour-function is defined in the following way [2]:



[2]Hue is a colour-function with values on a colour-spectrum. It is clear that in black-white print in this paper, which for example does not differentiate orange from blue, the distinction of visual data given by the function `Hue` is decreased. Even in such conditions the function `Hue` can be applied to show some properties of Gaussian and mean curvatures.

Example 1

The parametric equations of *an ellipsoid* with a center at the origin $O(0,0,0)$ are:

$x(u,v) = a\cos u \sin v,\ y(u,v) = b\sin u \sin v,\ z(u,v) = c\cos v,\ (u,v) \in [0,2\pi] \times (0,\pi)$

where $a,b,c \in \mathbb{R}^+$.

In Figure 1 we show the unit sphere with radii $a = b = c = 1$ (Fig. 1a), the oblate ellipsoid with axes lengths $a\!=\!b\!=\!4$, $c\!=\!1$ (Fig. 1b) and the ellipsoid with axes lengths $a\!=\!2.5$, $b\!=\!4$, $c\!=\!2$ (Fig. c) coloured by the function `Hue[3gcurvatue]` and the graphs of their Gaussian curvatures (Fig. $1a_1$, Fig. $1b_1$, Fig. $1c_1$).

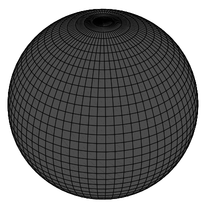


Figure 1a

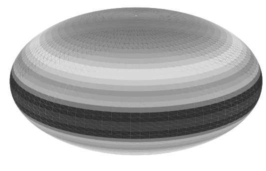


Figure 1b

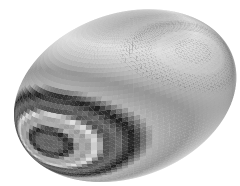


Figure 1c

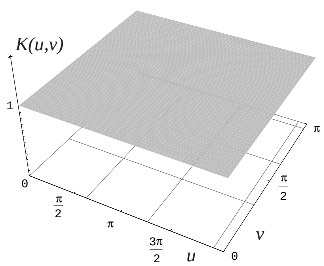


Figure $1a_1$

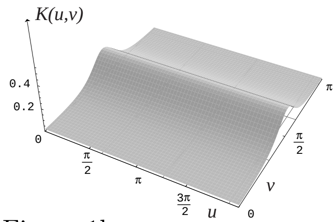


Figure $1b_1$

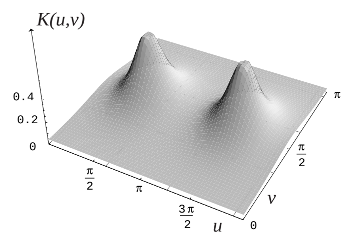


Figure $1c_1$

Example 2

The parametric equations of *the circular helicoid* are:

$x(u,v) = a\,v\cos u,\ y(u,v) = a\,v\cos u,\ z(u,v) = b\,u,$ where $a,b \in \mathbb{R}\setminus\{0\}$.

In Figure 2 we show the circular helicoid ($a\!=\!2$, $b\!=\!0.5$) over the domain $[-\frac{4}{3}\pi, \frac{17}{12}\pi] \times [-1,1]$ coloured by the functions `Hue[4gcurvature]` (Fig. 2a) and `Hue[mcurvature]` (Fig. 2c), as well as the graphs of its Gaussian (Fig. 2b) and mean (Fig. 2d) curvatures. It is clear from Fig. 2c and Fig. 2d that it is a *minimal* surface.

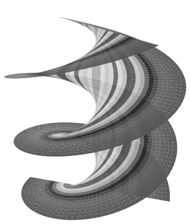


Figure 2a

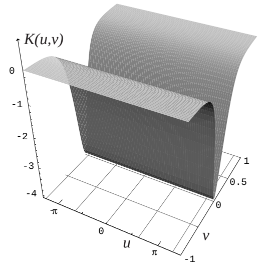


Figure 2b

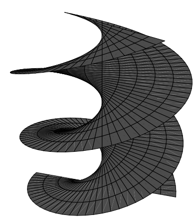


Figure 2c

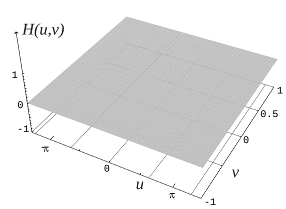


Figure 2d

Example 3

The parametric equations of the presented *hyperbolic paraboloid* are:

$x(u, v) = u,\ y(u, v) = v,\ z(u, v) = u^2 - v^2, \quad (u, v) \in [-1, 1] \times [-1, 1].$

In Figure 3 we show this paraboloid coloured by the functions `Hue[gcurvature]` (Fig. 3a) and `Hue[2mcurvature]` (Fig. 3c), as well as the graphs of its Gaussian (Fig. 3b) and mean (Fig. 3d) curvatures.

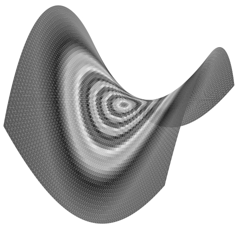


Figure 3a

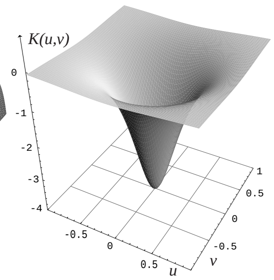


Figure 3b

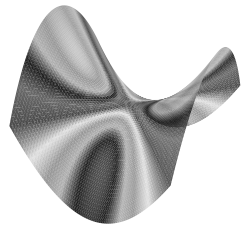


Figure 3c

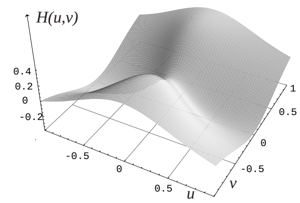


Figure 3d

Example 4

The parametric equations of the presented 3rd degree *parabolic conoid* are:

$x(u, v) = u,\ y(u, v) = v,\ z(u, v) = 0.5(uv^2 - 3v^2 - u + 3), \quad (u, v) \in [1, 5] \times [-2, 2].$

In Figure 4 we show this conoid coloured by the functions `Hue[gcurvature]` (Fig. 4a) and `Hue[mcurvature]` (Fig. 4c), as well as the graphs of its Gaussian (Fig. 4b) and mean (Fig. 4d) curvatures.

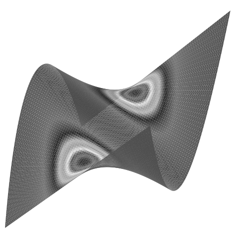


Figure 4a

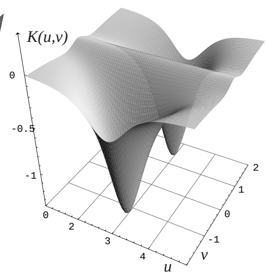


Figure 4b

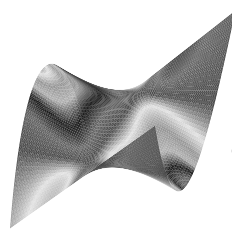


Figure 4c

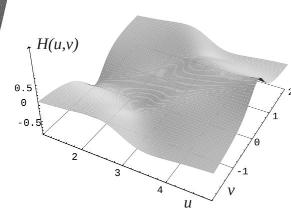


Figure 4d

Example 5

The parametric equations of the presented *monkey saddle* are:

$x(u, v) = u,\ y(u, v) = v,\ z(u, v) = u^3 - uv^2, \quad (u, v) \in [-0.8, 0.8] \times [-0.8, 0.8].$

In Figure 5 we show this monkey saddle coloured by the functions `Hue[gcurvature]` (Fig. 5a) and `Hue[2mcurvature]` (Fig. 5c), as well as the graphs of its Gaussian (Fig. 5b) and mean (Fig. 5d) curvatures.

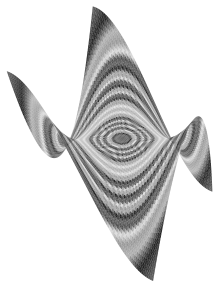


Figure 5a

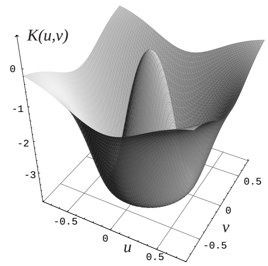


Figure 5b

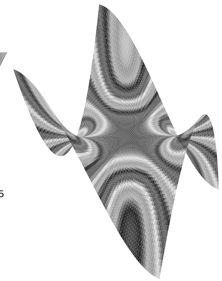


Figure 5c

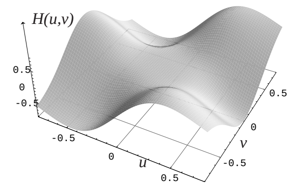


Figure 5d

Example 6

The parametric equations of the presented surface are:

$$x(u,v) = u,\ y(u,v) = v,\ z(u,v) = \sin u \sin v, \quad (u,v) \in [\frac{1}{2}\pi, \frac{5}{2}\pi] \times [-\frac{1}{2}\pi, \frac{3}{2}\pi].$$

In Figure 6 we show this surface coloured by the functions `Hue[gcurvature]` (Fig. 6a) and `Hue[mcurvature]` (Fig. 6c), as well as the graphs of its Gaussian (Fig. 6b) and mean (Fig. 6d) curvatures.

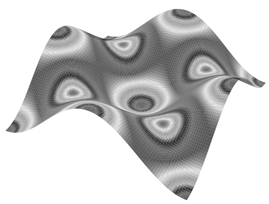


Figure 6a

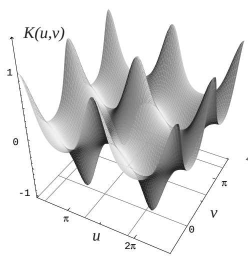


Figure 6b

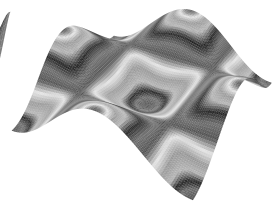


Figure 6c

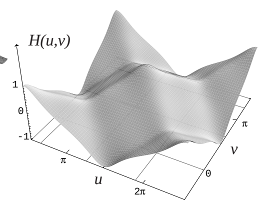


Figure 6d

3. *webMathematica*

As it is well known web is based on *client/server architecture*. When a user wants to see some web page on his browser (Internet explorer, Opera, Mozilla etc), the browser (client) sends requirement to the respective server to display that page. Then the server sends the content of the page to the client which shows the page to the user.

The language for designing web pages is Hyper Text Markup Language (HTML). This language does not support interactive solutions of mathematically formulated problems and interactive visualisations of results. To be able to solve mathematical problems it is necessary to use some of specialised programs, such as *Mathematica*. But the program *Mathematica* can not be directly activated by HTML. Therefore, HTML server has to be connected with *Mathematica* and it is done with the program *webMathematica*. In other words, *webMathematica* bridges web server and the program *Mathematica* which enables interactive calculations and visualisations on web pages.

The procedure of interactive communication is the following:

- A user feeds data for a certain mathematical problem into his client computer.

- The display of the page with results (numerical, symbolic or graphical) is required from web server.

- Web server through *webMathematica* activates the program *Mathematica* which produces results and forwards them to *webMathematica*. Then *webMathematica* sends the results to web server.

- Web server sends the page with the results to the client which displays web page to the user.

In order to be able to use *webMathematica*, it is necessary to install *webMathematica* and *Mathematica* on the computer with web server. The web pages on server have to be written in extended HTML which is defined by the rules of *webMathematica*.

In Fig. 7 we show the print-screen of web page powered by *webMathematica* which we designed within IT project mentioned in the introduction. A user can write his inputs in white rectangles. **Visualize ▸** is the command button to start interactive communication. The results (LiveGraphics3D on computer) are shown in Fig. 8.

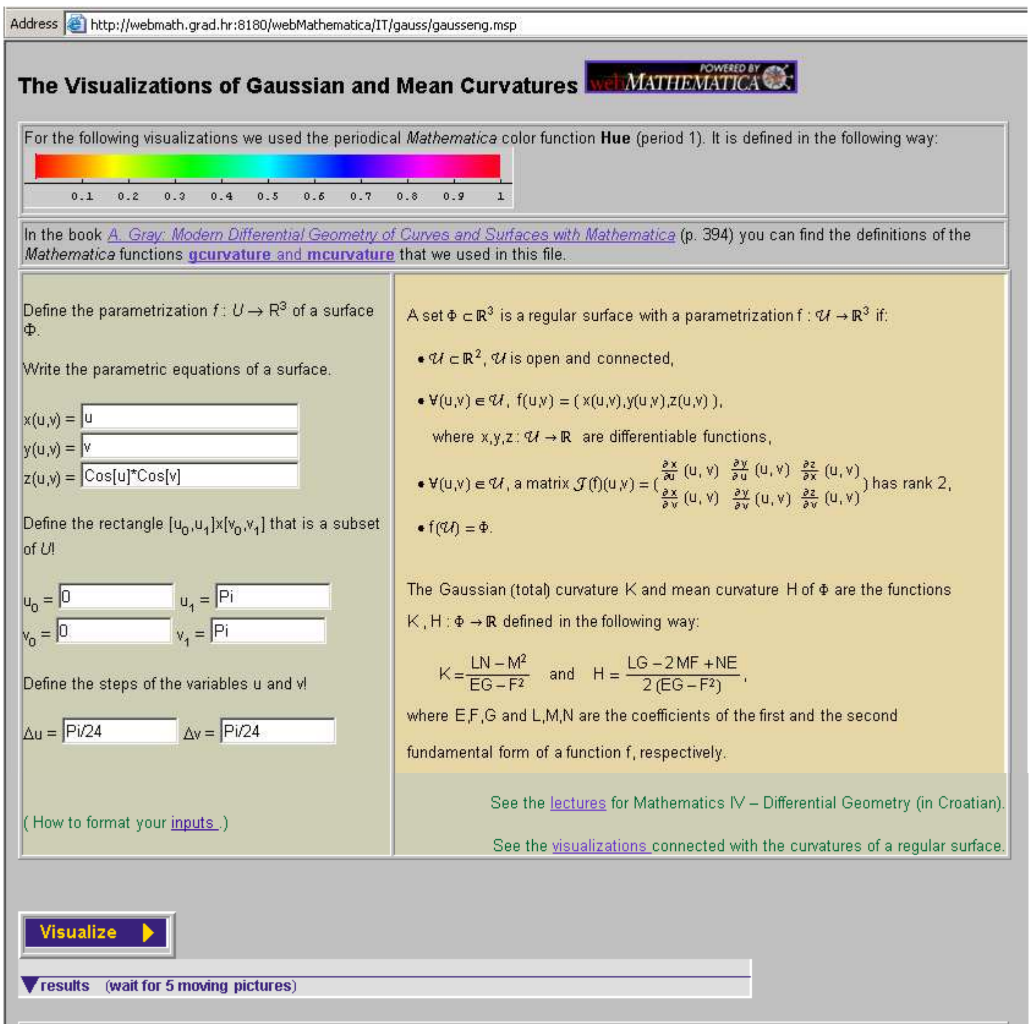


Figure 7

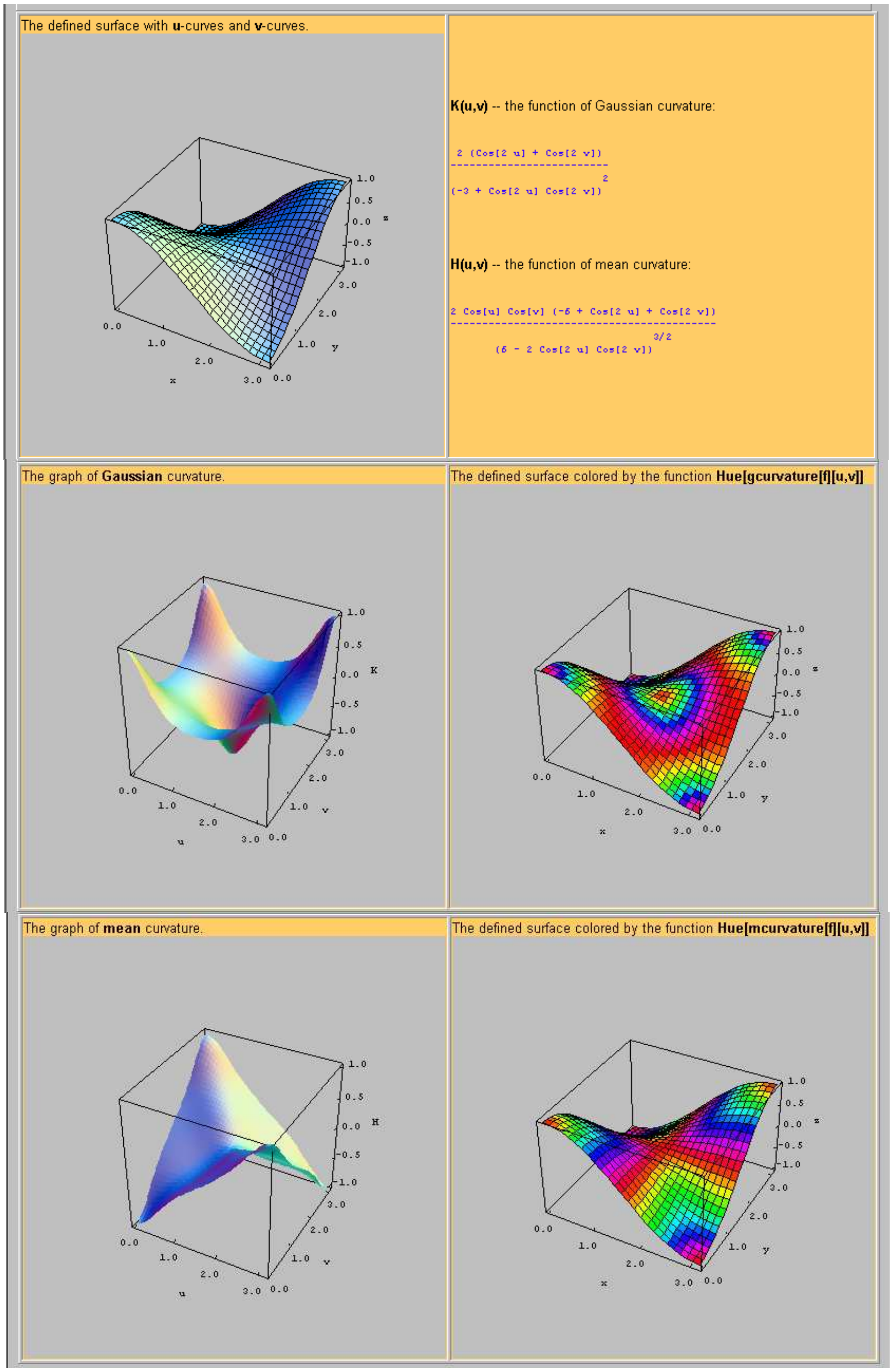


Figure 8: The print-screen of web page with results.

4. Conclusion

In teaching geometry *Mathematica* and can be used for designing diversified and high-quality educational material. Moreover, it is an ideal program for connecting the content of geometrical and mathematical subjects. New technology *webMathemaica* opens the door to interactive computing and visualization of data directly from the user's web provider.

We hope that the presented education material, created for the first year students, would improve students' understanding of the terms related to the normal, Gaussian and mean curvatures of a regular surface.

References

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