Some Examples of Using *Mathematica* and *webMathematica* in Teaching Geometry

Sonja Gorjanc

Faculty of Civil Engineering, University of Zagreb
Kačićeva 26, 10000 Zagreb, Croatia
email: sgorjanc@grad.hr

Abstract. The paper outlines some thoughts on teaching geometry at technical faculties, as well as various possibilities and advantages of the application of *Mathematica* in teaching, and gives some examples of the use of *Mathematica* visualizations in teaching geometry at the Faculty of Civil Engineering in Zagreb. The examples are extracts from the lectures, internet pages and sudents' projects on algebraic ruled surfaces and Gaussian and mean curvatures.

1. Introduction

At the technical faculties in Croatia geometry is primarily taught within the Descriptive geometry course in the first year. The way of teaching is very traditional. Computers are rarely used. There are many reasons for such situation which are probably similar to those in some other transitional countries : financial conditions, the policy of the university, teaching staff etc.

The author finds that the most significant positive effect of the traditional way of teaching is that the high level of geometric theory is preserved. On the other hand, there are negativne effects, such as students' declining interest and a lack of motivation and an increasing gap between the subjects of geometry and mathematics. Majority of students do not realize that the contents dealt with in geometrical and mathematical subjects are often the same but treated in different ways - constructive and analitical.

2. *Mathematica* and *webMathematica* in Teaching Geometry

In author's opinion, the main objective of our teaching is to develop cognitive and perceptive abilities and the knowledge of the students in the field of the threedimensional Euclidean space, as well as to enhance their knowledge of constructive methods necessary for the graphics communcation of future engineers. In the computorization of teaching, the development of constructive methods and skills is by no doubt connected with CAD programs. For that purpose *Mathematica* cannot help. However, it is very convenient for teaching the geometry

of Euclidean space becuse of its great graphics possibilities. Moreover, it is an ideal program for connecting the content of geometrical and mathematical subjects.

    Mathematica is a program which is used as a numerical and symbolical calculator, computer language, the system for visualization of functions and data, the platform for designing packages for specific applications and creating interactive documents with text, animations and sound, etc. *Mathematica* cannot be directly activated by HTML, but *webMathematica* is a program which bridges web server and the program *Mathematica* and enables interactive calculations and visualizations on the web pages. The web pages on server have to be written in extended HTML which is defined by the rules of *webMathematica*. Such complex systems with very sophisticated graphics can be applied creatively in teaching geometry. They are especially suitable for developing educational materials. For example, for designing:

    - presentation notebooks and web applications with animations and sound,

    - web applications for interactive learning on the Internet,

    - educational notebooks used in computer classrooms.

3. Examples

Since the major advantage of the use of *Mathematica* and *webMathematica* in teaching geometry is that they allow the design of sophisticated animations as well as 3D pictures moved by mouse (*LiveGraphics3D*), a great number of geometric properties can be clearly visualized. Because of the nature of the medium of writing, the examples listed in the paper have rather poor quality compared to the original.

3.1. Lecture - hyperbolic paraboloid

Because of their geometric properties ruled surfaces are suitable for roofing, therefore they are especially interesting for the students of civil engineering. The second degree surfaces (the hyperboloid of one sheet and the hyperbolic paraboloid) are the simplest. Thay have two systems of rulings, i.e. through each point on such surface two rulings pass - one from each system. All rulings of one system are skew and each of them cuts all rulings of the other system. Any three rulings of one system can be chosen as directing curves. [1], [7], [6]

3.1.1. The generations of the hyperbolic paraboloid

The hyperbolic paraboloid $\mathcal{HP}$ is a ruled quadric with the simplest parametrization

$$(u, v) \rightarrow (u, v, uv). \tag{1}$$

It is a conoidal surface with two lines at infinity. Each system of rulings has one ruling at infinity. The rulings of $\mathcal{HP}$ can be constructed as shown in Fig. 1 and Fig 2.
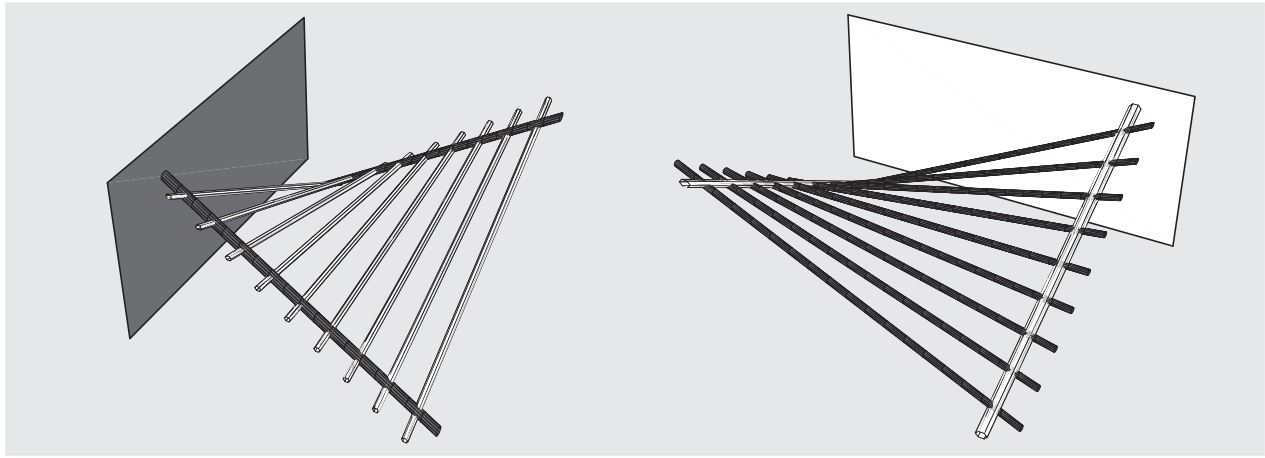
Figure 1: Each system of rulings of $\mathcal{HP}$ can be constructed as a system of lines which cut two skew directing lines and which are parallel with a directing plane - the plane which contains the ruling at infinity. (Directing lines are not parallel with the directing planes.)
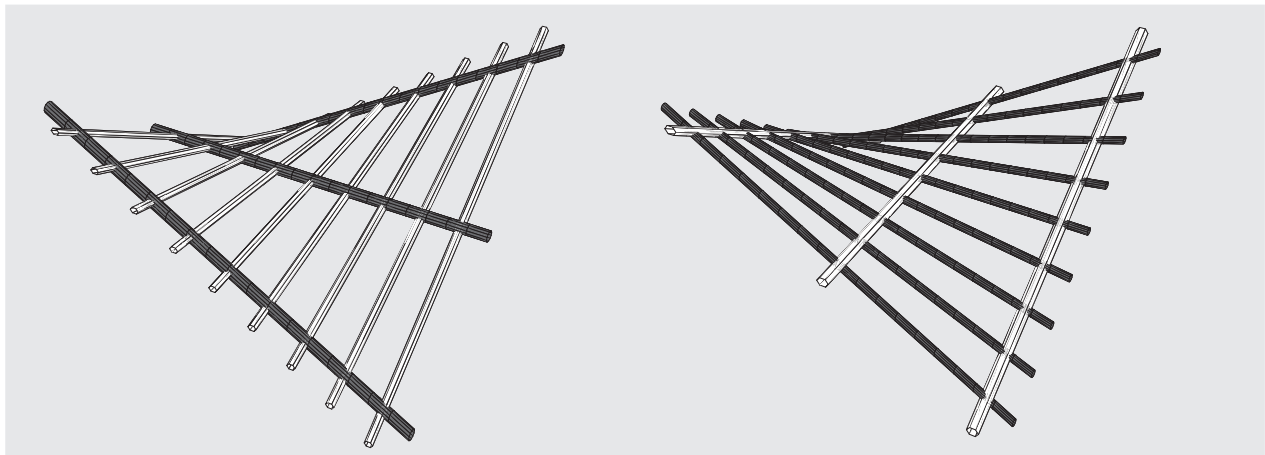

Figure 2: The directing curves are three skew lines parallel with the same plane. Any three rulings of one system can be chosen as directing curves.

The hyperbolic paraboloid is also a translation surface. It can be obtained by moving the parabola of one main section along the parabola of other main section. The other very simple parametrization of $\mathcal{HP}$ is

$$(u, v) \rightarrow (u, v, u^2 - v^2). \tag{2}$$
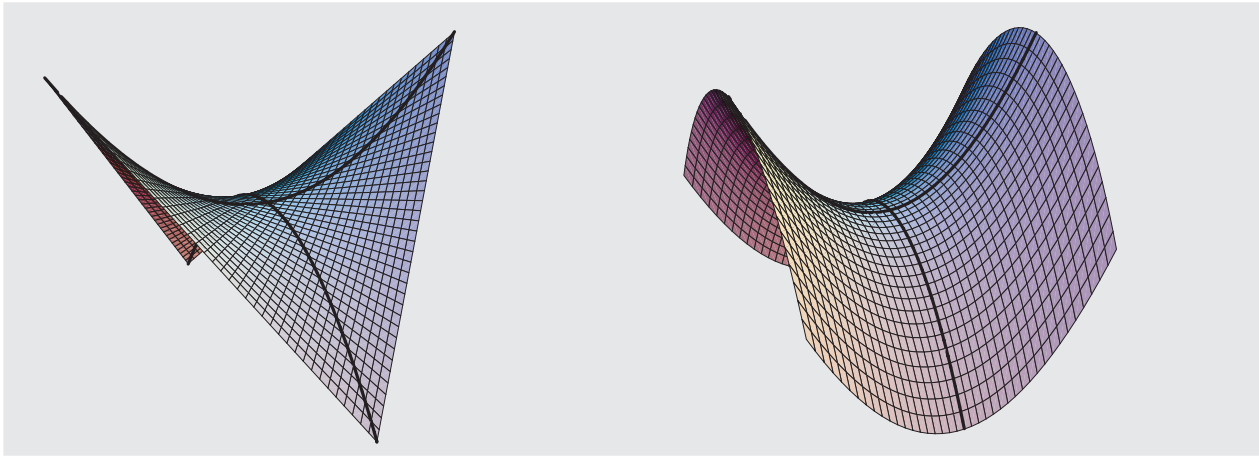
Figure 3: Two parabolas which generate $\mathcal{HP}$ as a translate surface. *Mathematica* graphics are obtained by the use of parametrizations (1) and (2).

3.1.2. Intersections of the hyperbolic paraboloid and planes

The tangent plane cuts $\mathcal{HP}$ into the pair of rulings (Fig. 4a). Since the plane at infinity contains two rulings it is a tangent plane at the point of the intersection of directing planes. Since $\mathcal{HP}$ has two real lines at infinity, the conic intersection of that surface and a plane possesses two or one real point at infinity. Therefore, there are no ellipses on $\mathcal{HP}$. A plane cuts $\mathcal{HP}$ into a parabola if it is parallel with the intersection of directing planes (Fig. 4b). In all other cases, excluding tangent planes, the intersections of $\mathcal{HP}$ are hyperbolas (Fig. 4c).

a   b   c
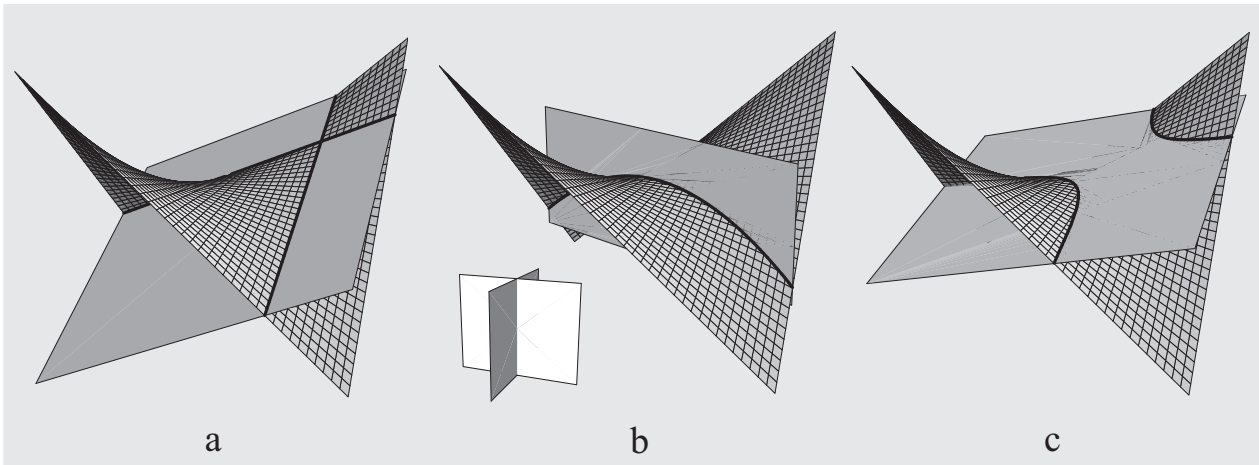
Figure 4:

3.1.3. Mean and Gaussian curvature

For future structural engineers it is important to have the knowledge of the Gaussian and mean curvatures. For example: Tensile fabric structure (e.g. membrane roof) in a uniform state of tensile prestress behaves like a soap film stretched over a wire which is bent in a shape

of a closed space curve. Soap film assumes a form which has the minimal area relative to all other surfaces stretched over the same wire, this surface is therefore called *minimal* surface. It can be shown that *mean curvature vanishes* at each point of that surface. $\mathcal{HP}$ is convinient for some visualization connected with Gaussian and mean curvatures.
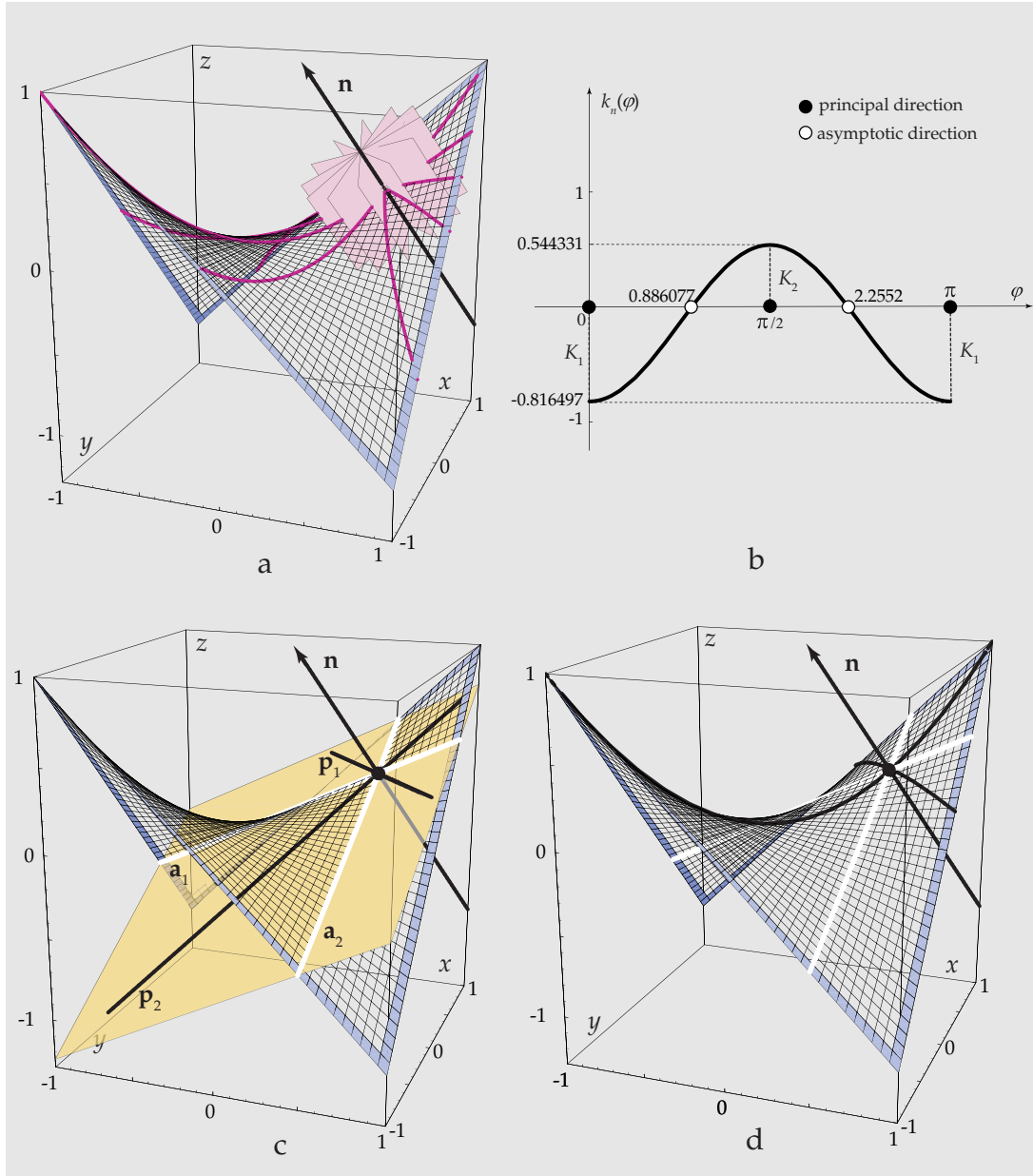

Figure 5: The pictures show part of the hyperbolic paraboloid defined by the parametirization (1), over the rectangle $[-1, 1] \times [-1, 1]$. The picture a shows six planes through a normal line and their intersections with the surface through the point $(0.5, 0.5, 0.25)$. The picture b shows the graph of the function of normal curvature in that point with principal and asymptotic directions. Asymptotic directions are defined by two rulings passing through the given point (for the parametrization (1) rulings are u-curve and v-curve. The principal directions are defined by lines symetric to the rulings c. The picture d shows the principal and asymptotic curves through the given point.

The Gaussian curvature is a product and a mean curvature is a half of the sum of principal curvatures at the regular point of a surface. For computing we use the first and the second fundamental form. In the book [3, pp. 394] we can find *Mathematica* definitions of the functions *mcurvature* and *gcurvature* which compute mean and Gaussian curvatures in each regular point of the surface given by parametric equations. These functions enable us to plot the graphs of the Gaussian and mean curvatures of regular surfaces and to paint surfaces with colours which depend on these curvatures. For the following visualizations we used the periodical *Mathematica* colour function Hue (period 1) with values on a colour-spectrum. It is clear that in black-white print in this paper, which for example does not differentiate orange from blue, the distinction of visual data given by the function Hue is decreased. Even in such conditions the function Hue can be applied to show some properties of Gaussian and mean curvatures.
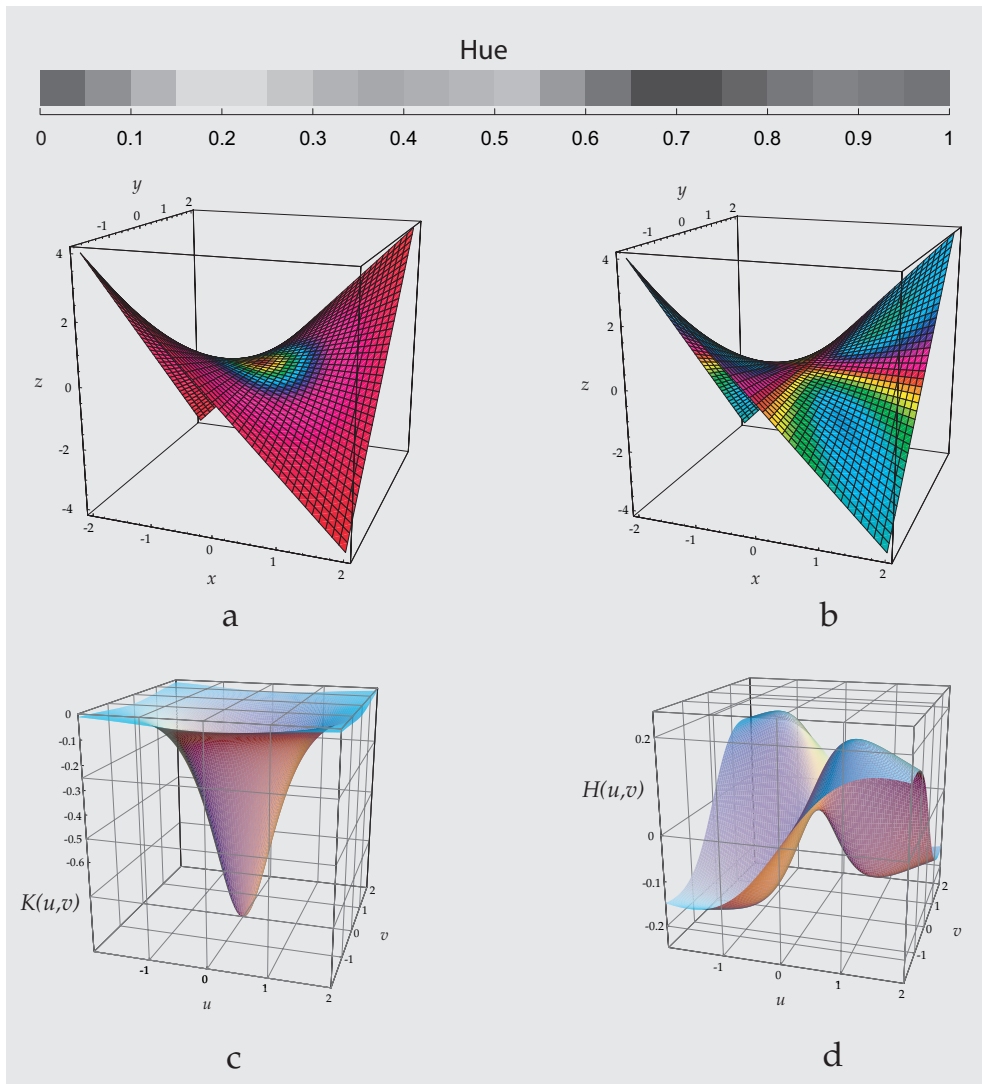
Figure 6: For the hyperbolic paraboloid defined by the parametrization (1) the figure shows: (*a*) The surface coloured by *Hue*(*gcurvature*). (*b*) The surface coloured by *Hue*(3*mcurvature*). (*c*) The graph of the function *gcurvature* over the rectangle $[-2, 2] \times [-2, 2]$. (*d*) The graph of the function *mcurvature* over the rectangle $[-2, 2] \times [-2, 2]$.

3.2. Internet pages

Internet pages without animations can be simply designed by saving *Mathematica* notebooks as HTML. If a notebook contains animations it would not be correctly saved in such a way, animations have to be exported from *Mathematica* and then mannualy included into HTML files. It is clear that graphics and animations exported from *Mathematica* can also be included into any file written in HTML. At the address *www.grad.hr/nastava/geometrija/ilustracije* one can find web pages with *Mathematica* graphics and animations which visualize generations and properties of conics, intersections of planes and cones or cylinders, intersections of cones and cylinders, ruled algebraic surfaces ([4], [5]), normal, Gaussian and mean curvatures...Although the author first designed these pages for students five years ago, the major part (especially files connected with curvatures) was created within IT project [1] (*http://www.grad.hr/itproject_math*). Within that project we also installed *webMathematica* on the web server at the Faculty of Civil Engineering in Zagreb and designed some interactive web pages powered by this program. In Fig. 7 and Fig. 8 we show the print-screen of the web page with visualizations of Gaussian and mean curvatures. A user writes his inputs in white rectangles and by pressing the command button Visualize ▶ starts interactive communication.

Figure 7:

The defined surface with **u**-curves and **v**-curves.

K(u,v) -- the function of Gaussian curvature:

$$\frac{2 \ (\text{Cos}[2 \ u] \ + \ \text{Cos}[2 \ v])}{(-3 \ + \ \text{Cos}[2 \ u] \ \text{Cos}[2 \ v])^2}$$

H(u,v) -- the function of mean curvature:

$$\frac{2 \ \text{Cos}[u] \ \text{Cos}[v] \ (-5 \ + \ \text{Cos}[2 \ u] \ + \ \text{Cos}[2 \ v])}{(5 \ - \ 2 \ \text{Cos}[2 \ u] \ \text{Cos}[2 \ v])^{3/2}}$$

The graph of **Gaussian** curvature.

The defined surface colored by the function **Hue[gcurvature[f][u,v]]**

The graph of **mean** curvature.

The defined surface colored by the function **Hue[mcurvature[f][u,v]]**

Figure 8: The results are *LiveGraphics3D* on computer.

3.3. Students' projects

Since some of our best students were very interested in working with the program *Mathematica*, the author encouraged them to explore issues beyond regular curriculum. The results of that collaboration are two students' papers which won the Rector's Prizes. The papers can be downloaded from the address *http://www.grad.hr/sgorjanc/teaching.html*.

3.3.1. Covering with parabolic conoid

In 2001 second-year students Sanja Filipan and Hrvoje Kvasnička wrote the paper *Covering with Parabolic Conoid*. Certain parts of it, extended with visualizations of Gaussian and mean curvatures, can be found in [2]. Students treated the class of 3rd degree parabolic conoids with directing curves p, l and d^∞ which can be connected with the Cartasian coordinate system as in Fig. 9, derived their equations and defined the functions for drawing parabolic conoids using *Mathematica* for each choice of the parameters a, b and c. Parabolic conoids can be used for covering in numerous ways. Various parts of conoids can be composed in simple series. Some ideas for the covering over the rectangular plan are given in Fig. 9.
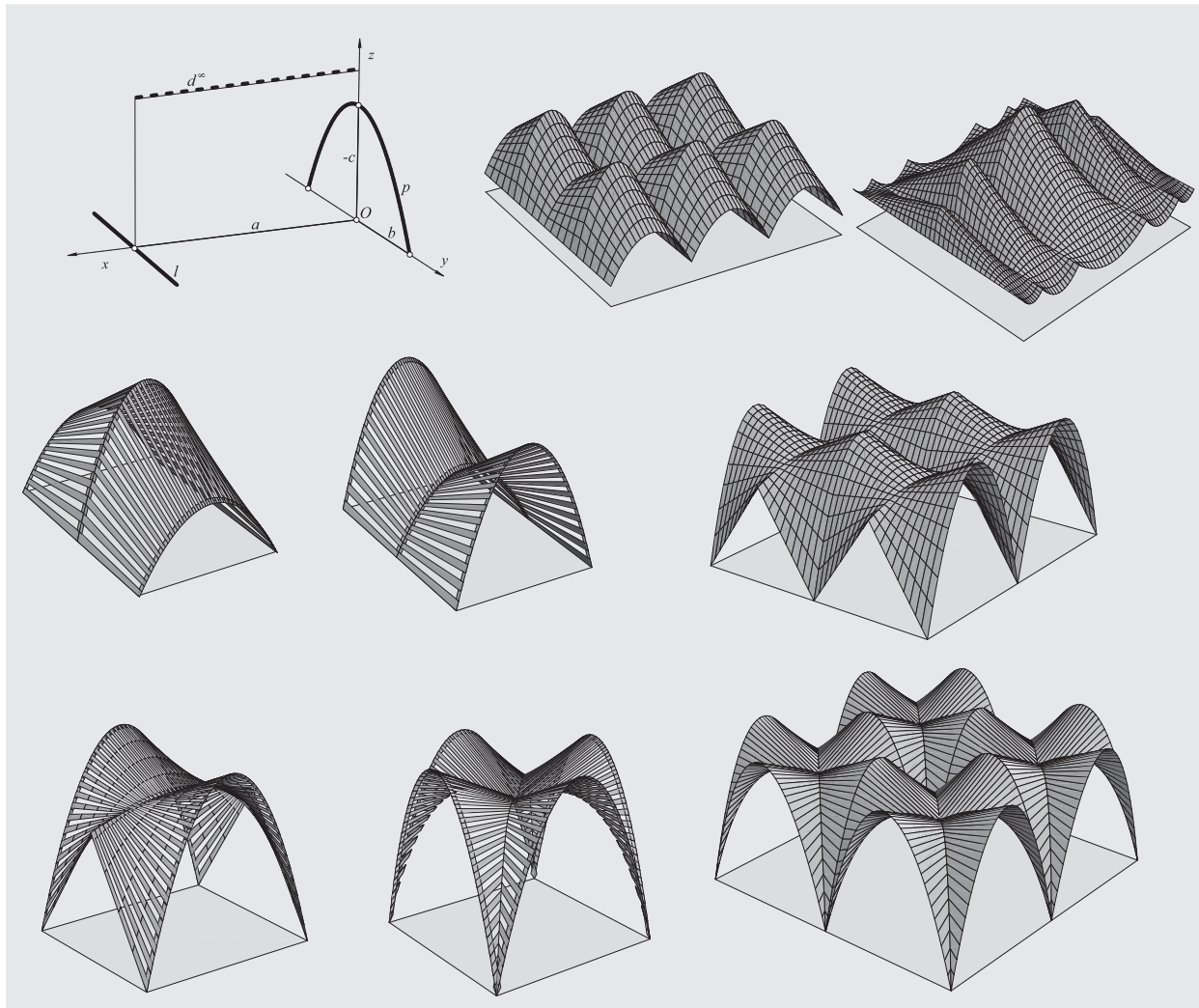

Figure 9:

3.3.2. Gaussian and mean curvatures of surfaces - visualizations with the program
   Mathematica

In 2004 third and fourth-year students Sanja Hak and Mario Uroš wrote the paper *Gaussian and Mean Curvatures of Surfaces - Visualizations with the Program Mathematica*. Thanks to their interest and talent, it was a pleasure to work with Sanja and Mario and the results they achieved were far beyond the expected. I enclose the summary of their paper together with several pictures.

   "In the *first* chapter elementary characteristics of tension structures are mentioned, as well as some specific problems during their design. A survey of photographies for this type of structures is given on the CD, which is enclosed in this paper. In the *second* and *third* chapter, using methods of analitic and differential geometry, with aid of the program Mathematica, the following terms are worked out: the parametrization of surfaces, the tangent plane and normal vector, the first and second fundamental form, principal and asymptotic directions, as well as the functions of Gaussian and mean curvatures with particular accent on ruled surfaces. In the *fourth* chapter, the visualizations of Gaussian and mean curvatures are given for eleven examples of surfaces. There is also an overview of the Mathematica notebook, which was used to obtain these visualizations. The program and input data for drawing are accessible on the enclosed CD. The *fifth* chapter contains definitions, as well as eleven examples of minimal surfaces. These examples are worked out through parametrization, calculation of principal curvatures and graphics in Mathematica. The file used for this treatment is also given on the enclosed CD. The *sixth* chapter contains a short description of numerical and experimental form finding of minimal nets."
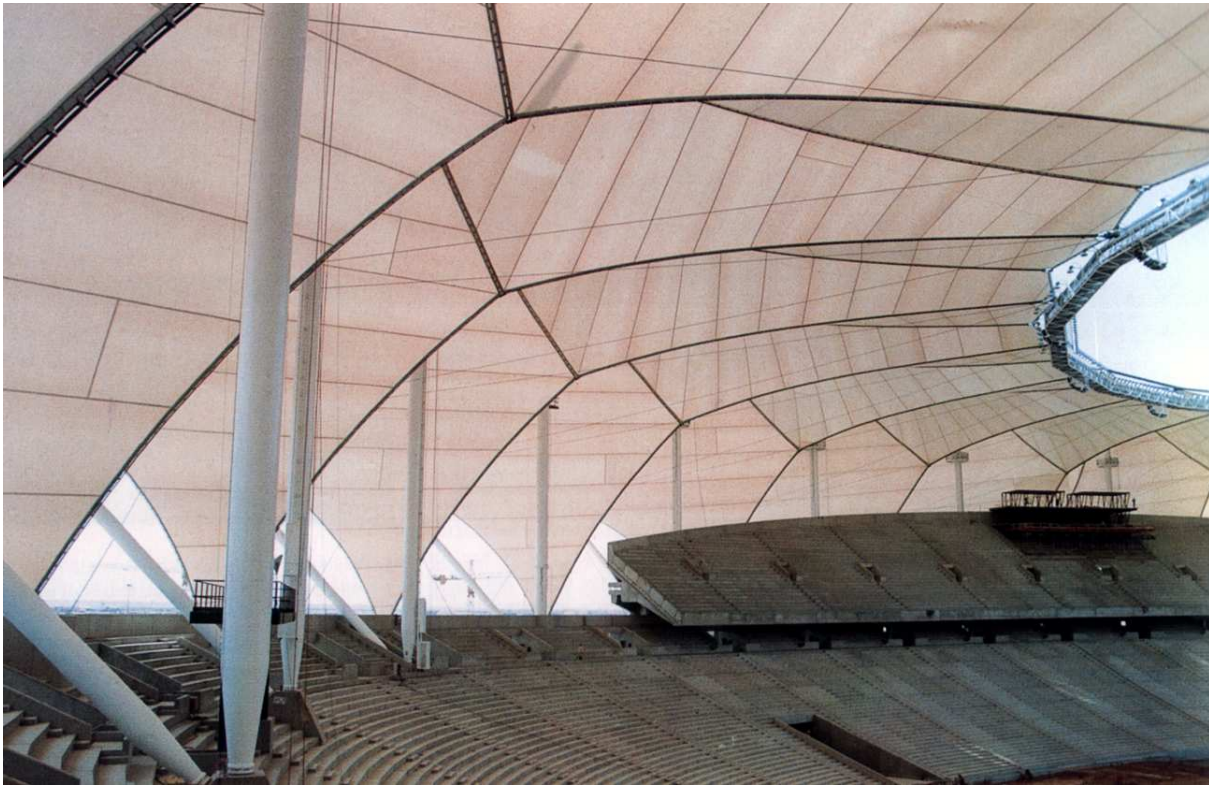

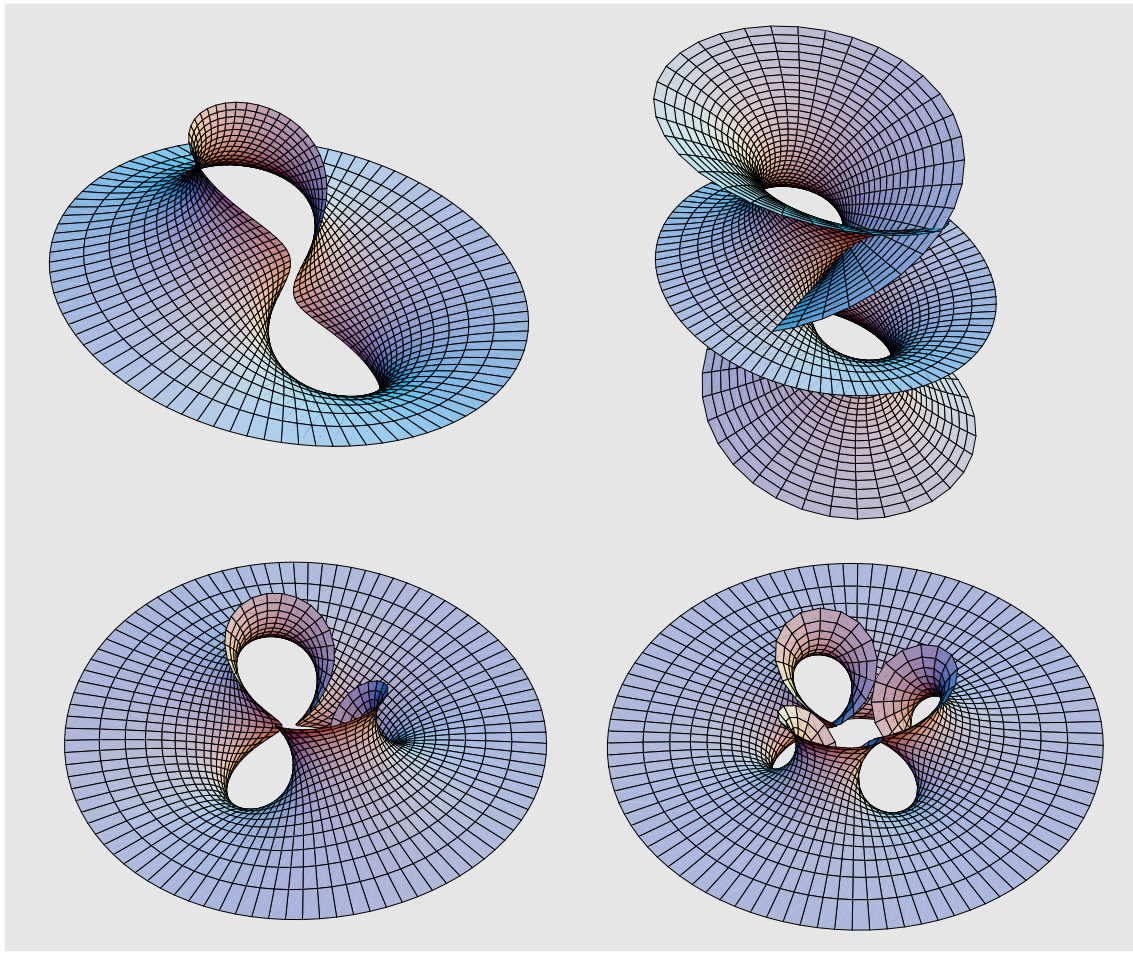Figure 10: Riyadh International Stadium, Riyadh, Saudi Arabia, 1985
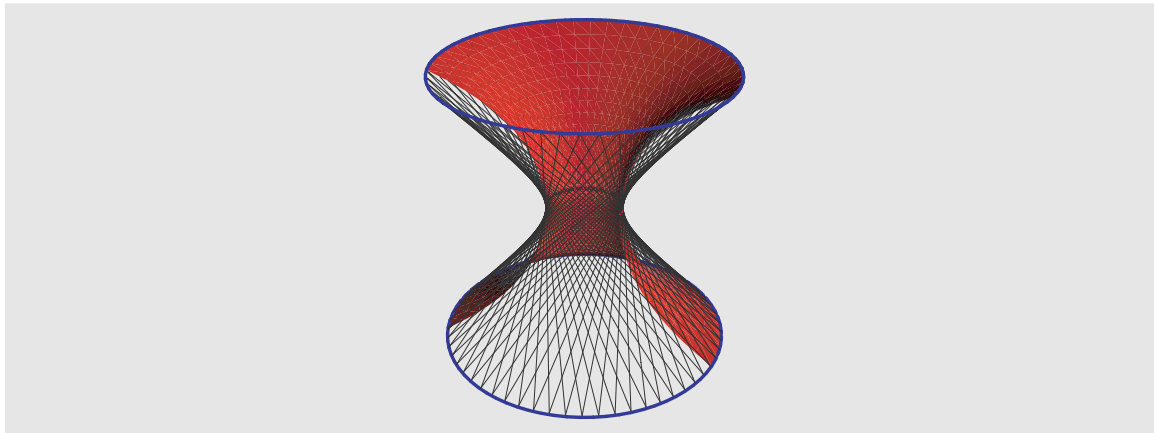
Figure 11: Richmond's minimal surfaces

Figure 12: Minimal surface (the catenoid) and minimal net (the hyperboloid of one sheet) bounded by the same parallel circles.

References

[1] H. Brauner, W. Kickinger: *Geometrija u graditeljstvu.* Školska knjiga, Zagreb, 1980.

[2] S. Filipan, S. Gorjanc, H. Kvasnička: *Natkrivanje paraboličkim konoidom.* KoG 5, 57-64 (2000).

[3] A. Gray: *Modern Differential Geometry of Curves and Surfaces with Mathematica.* CRC Press, Boca Raton, 1998.

[4] S. Gorjanc: *The Generation of Ruled Quartics in Mathematica.* Proceedings of Special SEFI European Seminar on Geometry in Engineering Education, Bratislava-Smolenice, Slovakia, 16-31 (1997).

[5] S. Gorjanc: *The Generation of Ruled Cubics by using Mathematica.* Proceedings of 8th ICGG, Austin, Texas, USA, 41-48 (1998).

[6] B. Kučinić , O. Kristoforović, I. Saler: *Oble forme u graditeljstvu.* Građevinar, Zagreb. 1992.

[7] V. Niče: *Deskriptivna geometrija II.* Školska knjiga, Zagreb. 1980.

Received