QUARTICS WITH MULTIPLE LINES IN E^3

Sonja Gorjanc
Faculty of Civil Engineering
Zagreb, Croatia

ABSTRACT

The paper gives a short review on seven classes of the quartics with multiple lines, points out some properties for each class and visualizes some examples with *Mathematica*.

1 INTRODUCTION

In this paper we use the term "quartic" for the 4th order surfaces in three dimensional Euclidean space. In the homogeneous Cartesian coordinates $(x{:}y{:}z{:}w)$, $x, y, z \in \mathbb{R}, w \in \{0, 1\}$, $(x{:}y{:}z{:}w)$ $\neq$(0:0:0:0), a quatric is given by the homogeneous equation $F^4(x, y, z, w) = 0$ of degree 4. In the language of synthetic geometry a quartic is a double infinite point system cut by any straight line at four points (imaginary points are included).

The purpose of this paper is to show a variety of quartics' forms.

2 QUARTICS WITH MULTIPLE LINES

According to the chapters on quartics in the books [10], [8], [9] and [7] the following kinds of multiple lines a quatric can possess are obtained:

- 1 triple straight line;
- 1 double twisted cubic;
- 1 double conic, 1 double straight line;
- 1 double conic;
- 3 double straight lines;
- 2 double straight lines;
- 1 double straight line.

2.1 Quartics with a triple straight line

A triple line is the highest singularity which a quartic can possess. The triple line is necessarily a straight line and a quatric is necessarily a ruled surface. Every plane section through the triple line consists of that line counted thrice and another straight line. If the triple line is the axis z $(x = 0, y = 0)$, the equation of the ruled quartic may be written in the form: $\mathbf{u_4} + z\mathbf{u_3} + w\mathbf{v_3} = 0$, where $\mathbf{u_4}, \mathbf{u_3}, \mathbf{v_3}$ are homogeneous polynomials in x and y of degree 4, 3, 3 respectively [10, p. 202].

In this paper we will relate to Sturm's classification of ruled quartics [9, pp. 246-294], [7, p. 80]. According to it there are four classes (IX, X, XI XII) of ruled quartics with a triple line. They differ by the number and kind of torsal lines.

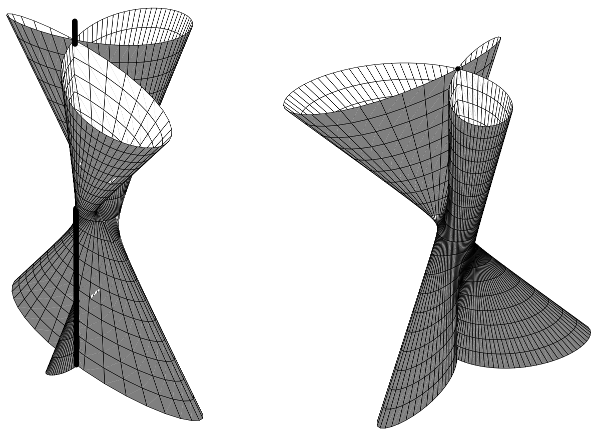

Figure 1: Two views of the ruled qurtic of the class IX. The directing curves are two non-intersecting conics and a triple line which cuts each conic at one point. The equations of the surface and its directing curves are given in [4, p. 63].

2.2 Quartics with a double twisted cubic

The quartics with a double twisted cubic are always ruled surfaces [10, p. 206]. They belong to classes III and IV and are reciprocals of the classes IX and X. They are quartics of zero deficency whose section by any plane is a quatric curve of deficency zero i. e. a plane quartic with three double points. The directing lines are a twisted cubic, one of cubic's chords and a straight line skew with the chord and the cubic.

2.3 Quartics with a double conic and a double straight line

They are also ruled quartics and belong to classes V and VI. The directing curves are two conics intersecting at two points and a straght line which cuts one of them.

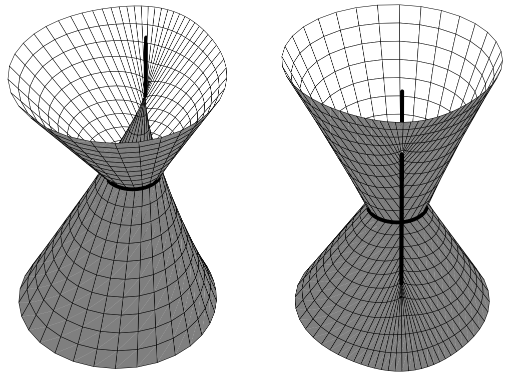

Figure 2: Two views of the ruled qurtic of the type V. The directing curves are two circles in the parallel planes and a double line which cuts a double circle. The equations of the surface and its directing curves are given in [4, p. 64].

2.4 Quartics with a double conic

If a double line of a quartic is one conic, the quartic is a non-ruled surface. A double conic can be real or imaginary. Cyclides (the most famous quartics) belong to that class and the absolute conic $(x^2 + y^2 + z^2 = 0, w = 0)$ is their double curve [8, pp. 1618-1628], [10, pp. 221-238], [1, pp. 201-250].

2.4.1 Dupin's cyclides

"Dupin's original method consisted of describing a cyclide as the envelope of all spheres tangent to three given spheres. Liouville showed that any cyclide of Dupin is the image under an inversion of a torus, circular cylinder or circular cone. Maxwell gave a direct construction of a cyclide of Dupin as the envelope of spheres whose centers move along ellipses, hyperbolas or parabolas:..." [5, p. 809]

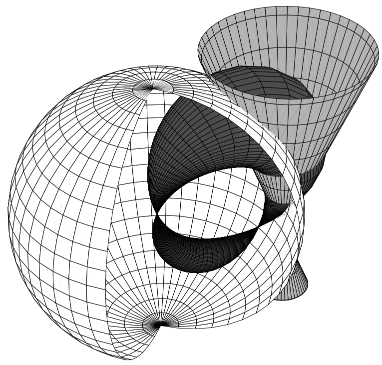

Figure 3: The picture shows the cyclide which is the image of a circular cone under the inversion with recpect to a sphere. The picture is obtained by using the function `invertedcircularcone` [5, p. 836].

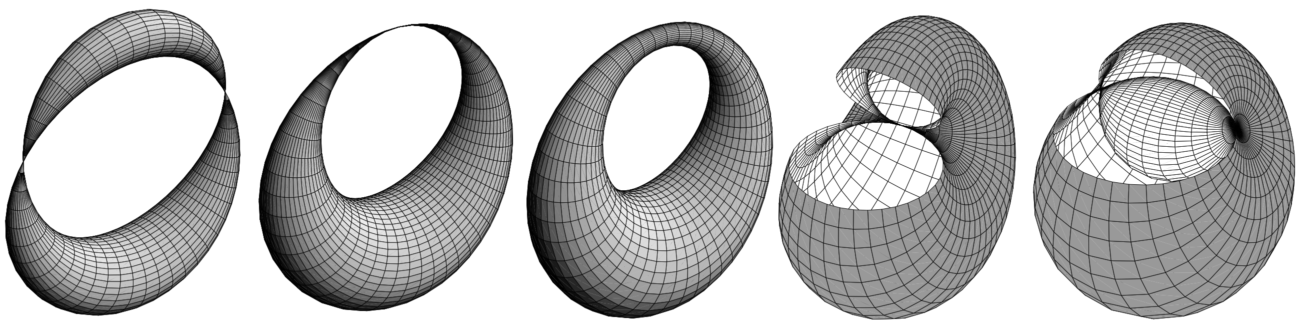

Figure 4: The picture shows Dupin's cyclides with an ellipse and a hyperbola as their focal set. The picture is obtained by using the function `ellhypcyclide` [5, p. 813].

2.4.2 Pedal surfaces of central quadrics

The locus of the feet of perpendiculars drawn from any fixed finite point P, called the pole, to the tangent planes of n-class surface Φ is the pedal surface of Φ for the pole P. It passes n times through the absolute conic and the pole P. The pedal surfaces of central quadrics are cyclides with a double point in the pole.

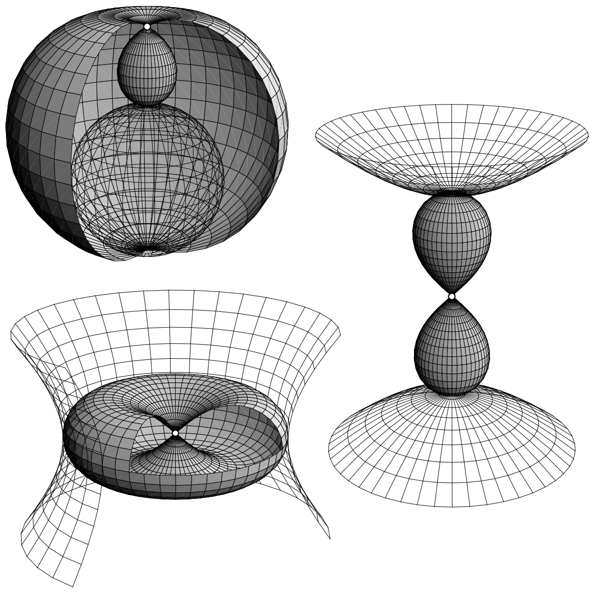

Figure 5: The pedal surfaces of a sphere, a hyperoloid of two sheets and a hyperboloid of one sheet. They are obtained as the surfaces of revolution of the pedal curves of a circle and a hyperbola.

2.4.3 Cyclides with a triple point

We describe here very simple analitical construction of cyclides with one triple point.

We use the proposition which can be found in [6, p. 251]:

If $X \subset \mathbb{R}^n$ is a hypersurface given by the polynomial $F(x_1, x_2, ..., x_n)$ and we write

$$F(x) = H_m(x) + H_{m+1}(x) + ... + H_n(x),$$

where $H_k(x)$ is homogeneous of degree k in $x_1, ..., x_n$; the tangent cone at the point (0,0,...,0) will be the cone of degree m given by the homogeneous polynomial H_m.

Now, homogeneous equation of each cyclide with a triple point in (0:0:0:1) can be written in the following form:

$$(x^2 + y^2 + z^2)^2 + w\mathbf{u_3} = 0,$$

where $\mathbf{u_3} = 0$ is homogeneous equation in x, y and z of degree 3 and represents the tangent cone in the triple point (0:0:0:1).

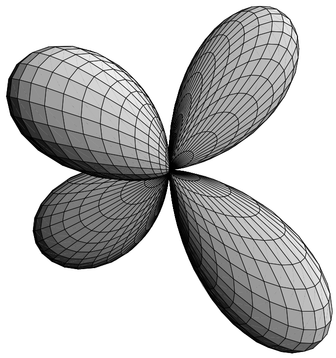

Figure 6: The cyclide with a homogeneous equation $(x^2 + y^2 + z^2)^2 + 2xyz = 0$. The third degree tangent cone at (0:0:0:1) breaks up into three real and different planes $(x = 0, y = 0, z = 0)$. If we use the polar coordinates we can obtain the following parametrization of the surface (covinient for drawing in *Mathematica*):

$$x(u, v) = \cos v \sin^3 v \sin 2u \cos u$$
$$y(u, v) = \cos v \sin^3 v \sin 2u \sin u$$
$$z(u, v) = -\cos v \sin^2 v \sin 2u$$

2.5 Quartics with three double straight lines

A quartic can possess three double straight lines in two cases.

2.5.1 Ruled quartics with three double lines

The ruled quartics with three double lines belong to the class VII. The directing curves are a conic and two straight lines without common points. Besides two skew double directing straight lines, these surfaces possess one double ruling which joins the intersection points of directing straight lines with the plane of a directing conic.

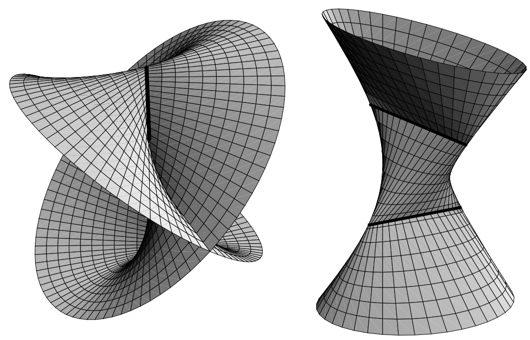

Figure 7: Two examples of ruled quartics with three double straight lines. The first surface (4th degree Plücker's conoid) possesses a double directing line at infinity which lies in the horizontal plane. The second surface contains a double ruling at infinity in the horizontal plane.

2.5.2 Stener's quartics

Steiner's quartics are non-ruled quartics with three double straight lines intersecting in one point. They are the reciprocals of a four-nodal cubic surfaces [10, p. 213]. If the coordinate axes are double lines of Steiner's quartic, its homogeneous equation can be written in the following way:

$$ay^2z^2 + bz^2x^2 + cx^2y^2 + dxyzw = 0.$$

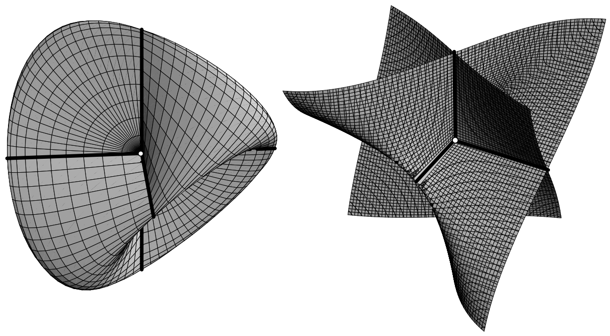

Figure 8: Two examples of Steiner's quartics. The first is Stainer's Roman surface, the most famous quartic of this class [5, p. 331]. Its equation is

$$y^2z^2 + z^2x^2 + x^2y^2 - 2xyzw = 0.$$

The equation of the second surface is

$$y^2z^2 - z^2x^2 + x^2y^2 - 2xyzw = 0.$$

2.6 Quartics with two double straight lines

The quatrics with two double straight lines are always ruled surfaces. They belong to the classes I, II and VIII. For the classes I and II the directing curves are the plane cubic with deficency one and two straight lines whose each line cuts the cubic. There are no conics on these quartics. The class VIII arises from the ruled quartics of the class VII if two directing double lines coincide.

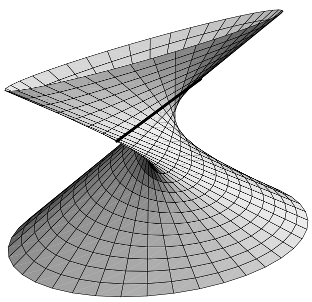

Figure 9: The ruled quartic of the class VIII which contains a finite double line and a double ruling at infinity in the horizontal plane. The equations of the surface and its directing curves are given in [4, p. 62].

2.7 Quartics with a double straight lines

If a double line of a quartic is one straight line, the quartic is a non-ruled surface which contains simple infinite conic system in the planes through a double line. In eight planes through a double line conics break up into two lines, thus there are sixteen simple lines on these surfaces.

2.7.1 Pedal surfaces of (1,2) congruences

The locus of the feet of perpendiculars drawn from any fixed finite point P, called the pole, to the rays of an (n, m)-congruence is the pedal surface of this congruence for the pole P. If the congruence is 1st order and 2nd class, the pedal surface is a quartic with a double straight line and contains absolute conic as a simple curve. Besides the absolute conic, these surfaces contain a pair of lines at infinity. The classification of these surfaces, accodring to the number of real simple lines at infinity and the number and kind of their singular points, are given in [2].

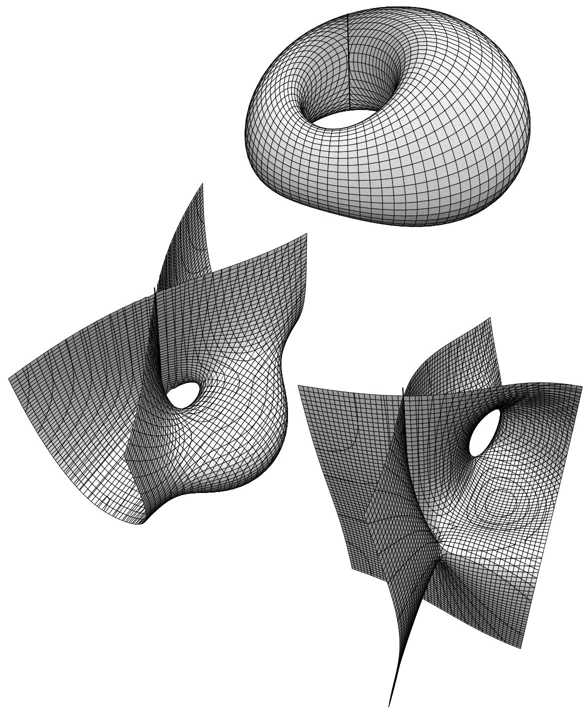

Figure 10: Types I_7, III_5 and V_8 of the pedal surfaces of (1,2) congruences [2].

2.7.2 Qurtics with a double straight line and a triple point

In the same way as we constructed cyclides with a triple point, we can construct the quartic with a double line and a triple point. They are given by equation:

$$\mathbf{u_4} + w\mathbf{v_3} = 0,$$

where $\mathbf{u_4}$ is homogeneous polynomial in x and y while $\mathbf{v_3} = 0$ is homogeneous equation in x, y and z and represents the tangent cone in the triple point (0:0:0:1). The equations

$\mathbf{u_4} = 0, w = 0$ represent the section with the plane at infinity which breaks up into a conic and a pair of lines through the point (0:0:1:0).

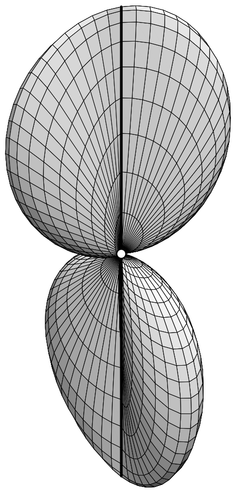

Figure 11: The surface represented by the homogeneous equation

$$(x^2 + y^2 + z^2)(x^2 + y^2) + 2z(x^2 - y^2) = 0.$$

The third degree tangent cone at (0:0:0:1) breaks up into three real and different planes $(x = y, x = -y, z = 0)$. If we use the polar coordinates we can obtain the following parametrization of the surface:

$$x(u,v) = \cos u \cos 2u \sin v$$
$$y(u,v) = \sin u \cos 2u \sin v$$
$$z(u,v) = \cos 2u(\cos v - 1)$$

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ABOUT THE AUTHOR

Sonja Gorjanc, Ph.D. is a senior lecturer in the Department of Mathematics, Faculty of Civil Engineering, University of Zagreb. Her research interest are in Projective and Euclidean geometry, *Mathematica* computer graphics, Curricular Developments in Geometry. She can be reached by e-mail: sgorjanc@grad.hr, by fax: +1-385-6600-642 or through postal address: Faculty of Civil Engineering, Kačićeva 26,10000 Zagreb, Croatia.