# Special Quartics with Triple Points

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**Abstract**

This paper deals with the special class of 4th order surfaces in 3-dimensional Euclidean space. The surfaces of that class contain the absolute conic, a double straight line and triple points. It is shown that such surfaces may contain at least two real triple points on a double line which are analysed according to the type of their tangent cones. The selected examples of the surfaces are drawn with the program Mathematica 4.1.

## 1 Introduction

In this paper we use the term “quartic” for the 4th order surfaces in three dimensional Euclidean space. In the homogeneous Cartesian coordinates $(x{:}y{:}z{:}w)$, $x, y, z \in \mathbb{R}$, $w \in \{0, 1\}$, $(x{:}y{:}z{:}w) \neq (0{:}0{:}0{:}0)$, a quatric is given by the homogeneous equation $F^4(x, y, z, w) = 0$ of degree 4. In the 19th and at the beginning of the 20th century their properties were studied intensively in a number of geometric books and papers. We present here the basic classification of those quartics which contain singular lines [7, vol. II, p. 200-252], [6, p. 1537-1787].

- quartics with a triple straight line, (the class contains only ruled quartics);
- quartics with a double twisted cubic, (the class contains only ruled quartics);
- quartics with a double conic section (the class contains cyclides);
- quartics with a double conic section and a double line, (the class contains only ruled quartics);
- quartics with three double lines, (the class contains Steiner's quartics);
- quartics with two double straight lines, (the class contains only ruled quartics);
- quartics with one double straight line, (the class contains ruled quartics, the pedal surfaces of (1,2) congruences [2] and the surfaces which we will be considered in this paper).

## 2 Tangent cones at singular points

The point $T(x_0{:}y_0{:}z_0{:}w_0)$ is a singular point of the surface given by the equation $F^n(x, y, z, w) = 0$ if and only if $F^n(x_0, y_0, z_0, w_0) = 0$ and $\frac{\partial F}{\partial x}(T) = \frac{\partial F}{\partial y}(T) = \frac{\partial F}{\partial y}(T) = \frac{\partial F}{\partial w}(T) = 0$. At such a point the tangents of the surface form a tangent cone of order equal to its multiplicity.

We state here some analytical properties of tangent cones, according to [7, Vol.I, p. 56] and [5, str. 251], which will be used in the proofs in the following section.

**Proposition 1** *Each homogeneous equation in $x$, $y$, $z$, of degree $n$, $H^n(x, y, z) = 0$, represents a cone $\mathcal{C}_O^n$ with a vertrex* (0:0:0:1). If the polynomial $H^n(x, y, z)$ is irreducible over the field $\mathbb{R}$, the cone $\mathcal{C}_O^n$ is a proper cone of order $n$. If $H^n(x, y, z) = H^{n_1}(x, y, z) \cdot H^{n_2}(x, y, z) \cdots H^{n_k}(x, y, z)$, $n_1 + n_2 + \cdots + n_k = n$ and $k \leqslant n$, the cone $\mathcal{C}^n(O)$ degenerates into the cones $\mathcal{C}^{n_1}(O)$, $\mathcal{C}^{n_2}(O), \ldots, \mathcal{C}^{n_k}(O)$.

**Proposition 2** *Each homogeneous equation in $x$, $y$ of degree 2, $H^2(x, y) = 0$, represents the pair of planes through the axis $z$. If $H^2(x, y)$ is reducible over the field $\mathbb{R}$ the planes are real, they coincide if $H^2(x, y)$ is a total square, and they are a pair of imaginary planes if $H^2(x, y)$ is irreducible over the field $\mathbb{R}$.*

**Proposition 3** *If $X \subset \mathbb{R}^n$ is a hypersurface given by the polynomial $F(x_1, x_2, ..., x_n)$ and we write*

$$F(x) = H_m(x) + H_{m+1}(x) + ... + H_n(x),$$

*where $H_k(x)$ is homogeneous of degree $k$ in $x_1, ..., x_n$; the tangent cone at the point* (0,0,...,0) will be the cone of order $m$ given by the homogeneous polynomial $H_m$.

## 3 Triple points on non ruled quartics through the absolute conic and with a double straight line

In the homogeneous Cartesian coordinates each quartic $\mathcal{F}$ with the double line $z$ ($x = 0, y = 0$) which contains the absolute conic $\omega$ ($x^2 + y^2 + z^2 = 0, w = 0$) can be presented by the following equation:

$$F(x, y, z, w) = (x^2 + y^2 + z^2)\mathbf{a_2} + w\mathbf{d_3} + zw\mathbf{b_2} + w^2\mathbf{c_2} = 0, \tag{1}$$

where $\mathbf{a_2}$, $\mathbf{b_2}$, $\mathbf{c_2}$, $\mathbf{d_3}$ are homogeneous polynomial in $x$, $y$ of degree 2, 2, 2 and 3, respectively, with real cofficients and at least one of the coefficients in polynomial $\mathbf{a_2}$ is different from 0, i.e. $\mathbf{a_2} \neq \mathbf{0}$.

The proof of this is a direct consequence of the equation of the quartic with the double line $z$ [7, vol. II, p. 217] and the fact that the section of such quartic and the plane at infinity is the absolute conic $\omega$ and the pair of lines through the point (0:0:1:0) given by the equations $\mathbf{a_2} = 0, w = 0$. Such proof can be found in [3, p. 136].

**Theorem 1** *$T$ is a triple point on the surface $\mathcal{F}$, given by the equation* (1), if and only if

$$T(0 : 0 : t : 1), t \in \mathbb{R}, \tag{2}$$

$$t^2\mathbf{a_2} + t\mathbf{b_2} + \mathbf{c_2} = 0, \tag{3}$$

$$\mathbf{d_3} + \tilde{z}(2t\mathbf{a_2} + \mathbf{b_2}) \neq \mathbf{0}. \tag{4}$$

*Proof:* The triple point of the surface $\mathcal{F}$ always lies on the double line $z$. (If a triple point did not lie on the double line, any line joining that point and the point on the double line would cut the surface into five points which is impossible for non ruled quartic.) Furthermore, the point (0:0:1:0) can not be a triple point of the surface $\mathcal{F}$. (If (0:0:1:0) was a triple point, it would be a triple point of any plane section through it, but the plane at infinity cuts the surface $\mathcal{F}$ into the degenerated 4th order curve $((x^2 + y^2 + z^2)\mathbf{a_2} = 0, w = 0)$ with a double point (0:0:1:0).) Therefore, the coordinates of triple points are always $(0{:}0{:}t{:}1)$.

If we translate the Cartesian coordinate system $O(x, y, z)$ into the system $T(\tilde{x}, \tilde{y}, \tilde{z})$, where $\tilde{x} = x, \tilde{y} = y, \tilde{z} = z - t$, we obtain the equation

$$F(\tilde{x}, \tilde{y}, \tilde{z}, 1) = (\tilde{x}^2 + \tilde{y}^2)\mathbf{a_2} + \mathbf{d_3} + (\tilde{z} + t)^2\mathbf{a_2} + (\tilde{z} + t)\mathbf{b_2} + \mathbf{c_2} = 0. \tag{5}$$

If $T$ is a triple point, then according to Proposition 3 a minimal degree of the homogeneous polynomials in (5) is 3 and it follows that $t^2\mathbf{a_2} + t\mathbf{b_2} + \mathbf{c_2} = 0$ and $\mathbf{d_3} + \tilde{z}(2t\mathbf{a_2} + \mathbf{b_2}) \neq \mathbf{0}$.
On the other hand, if $T(0{:}0{:}t{:}1)$, $t^2\mathbf{a_2} + t\mathbf{b_2} + \mathbf{c_2} = 0$ and $\mathbf{d_3} + \tilde{z}(2t\mathbf{a_2} + \mathbf{b_2}) \neq \mathbf{0}$, then the tangent cone at the point $T(0{:}0{:}t{:}1)$ is a cone of 3rd order, i.e. $T$ is a triple point of the surface $\mathcal{F}$. □

**Corollary 1** *If there are two different triple points on the double line of the surface $\mathcal{F}$, then the tangent planes at other points on the double line are given by the equation $\mathbf{a_2} = 0$.*

*Proof:* Without loss of generality we assume that $T_1(0{:}0{:}0{:}1)$ and $T_2(0{:}0{:}t{:}1)$, $t \neq 0$, $t \in \mathbb{R}$ are triple points of the surface $\mathcal{F}$. Then from Theorem 1 we conclude

$$\mathbf{c_2} = \mathbf{0} \quad \text{and} \quad \mathbf{b_2} = -t\mathbf{a_2}. \tag{6}$$

Let $Z_0(0{:}0{:}z_0{:}1)$ be the point on the double line different from $T_1$ and $T_2$, i.e. $z_0 \neq 0$ and $z_0 \neq t$. In the coordinate system $Z_0(x', y', z')$, $x' = x$, $y' = y$, $z' = z - z_0$, the surface $\mathcal{F}$ is given by the equation:

$$F(x', y', z', 1) = (x'^2 + y'^2 + z'^2)\mathbf{a_2} + \mathbf{d_3} + z'(2z_0 - t)\mathbf{a_2} + z_0(z_0 - t)\mathbf{a_2} = 0. \tag{7}$$

According to the Proposition 3 $z_0(z_0 - t)\mathbf{a_2} = 0$ is the equation of the tangent cone at the binode $Z_0$. Since $z_0(z_0 - t) \neq 0$, it follows that $\mathbf{a_2} = 0$ is the equation of a tangent cone at any point different from $T_1$ and $T_2$ on a double line. □

See: Fig. 1-3

**Theorem 2** *If $T(0{:}0{:}t{:}1)$ is a triple point of the surface $\mathcal{F}$ given by the equation* (1), then for the tangent cone $\mathcal{TC}_{(T,\mathcal{F})}$ of the surface $\mathcal{F}$ at the point $T$ one of the following facts is valid:

1. $\mathcal{TC}_{(T,\mathcal{F})}$ is a proper (non degenerate) cone of 3rd order if and only if $\mathbf{d_3} + 2\tilde{z}t\mathbf{a_2} + \tilde{z}\mathbf{b_2}$ is irreducible over the field $\mathbb{R}$.

2. $\mathcal{TC}_{(T,\mathcal{F})}$ degenerates into the real 2nd order cone and a plane if and only if the polynomials $\mathbf{d_3}$ and $2t\mathbf{a_2} + \mathbf{b_2}$ have a common linear factor into $x$ and $y$.

3. $\mathcal{TC}_{(T,\mathcal{F})}$ degenerates into three (independent) planes intercesting into the point $T$ if and only if $2t\mathbf{a_2} + \mathbf{b_2}$ is the factor of $\mathbf{d_3}$ or $\mathbf{d_3} = 0$.

4. $\mathcal{TC}_{(T,\mathcal{F})}$ degenerates into three planes through the double line $z$ if and only if $2t\mathbf{a_2} + \mathbf{b_2} = 0$ and $\mathbf{d_3} \neq 0$.

*Proof:* According to the Theorem 1, equation (5) and Proposition 3, with respect to the coordinate system $T(\tilde{x}, \tilde{y}, \tilde{z})$ the tangent cone of the surface $\mathcal{F}$ at the triple point $T(0{:}0{:}t{:}1)$ is given by

$$\mathbf{d_3} + \tilde{z}(2t\mathbf{a_2} + \mathbf{b_2}) = 0. \tag{8}$$

The axis $z(x = 0, y = 0)$ is the double line of the tangent cone given by the equation (8). Namely, each 3rd order cone with the double line $z$ and the vertrex (0:0:0:1) is given by the equation $\mathbf{u_3} + z\mathbf{u_2} = 0$, where $\mathbf{u_3}, \mathbf{u_2}$ are homogeneous polynomials in $x$ and $y$ of degree 3 and 2, respectively.

If the polynomial $\mathbf{d_3} + 2\tilde{z}(t\mathbf{a_2} + \mathbf{b_2})$ is irreducible over the field $\mathbb{R}$, the tangent cone $\mathcal{TC}_{(T,\mathcal{F})}$ is an irreducibile 3rd order cone with the double line $z$ (according to the Proposition 1).

If the polynomial $\mathbf{d_3} + \tilde{z}(2t\mathbf{a_2} + \mathbf{b_2})$ is reducible consisting of a linear and an (irreducible) 2nd order factor, we can obtain three different types of tangent cones because the greatest exponent of $\tilde{z}$ is 1.

- The 2nd order factor is homogeneous in $x$, $y$ and $z$, while the linear factor is homogeneous in $x$ and $y$. In this case $\mathbf{d_3}$ and $2t\mathbf{a_2} + \mathbf{b_2}$ have a common linear factor and the tangent cone degenerates into the plane through $z$ and an irreducibile, real cone of order 2 through $z$ (Proposition 1).

- The 2nd order factor is homogeneous in $x$ and $y$, while the linear factor is homogeneous in $x$, $y$ and $z$. In this case $2t\mathbf{a_2} + \mathbf{b_2}$ is a factor of $\mathbf{d_3}$ or $\mathbf{d_3} = 0$ and $2t\mathbf{a_2} + \mathbf{b_2}$ is irreducible over the field $\mathbb{R}$. The tangent cone $\mathcal{TC}_{(T,\mathcal{F})}$ degenerates into three planes. Two of them are imaginary planes through the line $z$ (Proposition 2), and the third is a real plane through $T$ which does not contain the line $z$.

- Both, the linear and the 2nd order factor are homogeneous in $x$ and $y$. This case occurs only if $2t\mathbf{a_2} + \mathbf{b_2} = 0$, $\mathbf{d_3} \neq 0$ with $\mathbf{d_3}$ reducible over the field $\mathbb{R}$ to a linear and irreducible factor of order 2. The cone $\mathcal{TC}_{(T,\mathcal{F})}$ degenerates into three planes through the line $z$ and the two of them are imaginary.

If the polynomial $\mathbf{d_3} + \tilde{z}(2t\mathbf{a_2} + \mathbf{b_2})$ is reducible to three linear factors, we can obtain two different types of a tangent cone.

- $2t\mathbf{a_2} + \mathbf{b_2}$ is a factor of $\mathbf{d_3}$ or $\mathbf{d_3} = 0$ and $2t\mathbf{a_2} + \mathbf{b_2}$ is reducible over the field $\mathbb{R}$. The tangent cone $\mathcal{TC}_{(T,\mathcal{F})}$ degenerates into three planes, two of them real (different or coinciding) through the line $z$ (Proposition 2) and the third plane not containing $z$.

- $2t\mathbf{a_2} + \mathbf{b_2} = 0$, $\mathbf{d_3} \neq 0$ and $\mathbf{d_3}$ is reducible over the field $\mathbb{R}$ to three linear factors. The cone $\mathcal{TC}_{(T,\mathcal{F})}$ degenerates into three real planes through the line $z$, whereby all three planes or just two of them may coincide. $\square$

See: Fig. 1-13

# 4 Examples

In this section we present thirteen pictures of the surfaces considered before, their triple points and tangent cones. The pictures are obtained by Mathematica 4.1. For each surface its equation in the Cartesian coordinates $(x, y, z)$ is given.

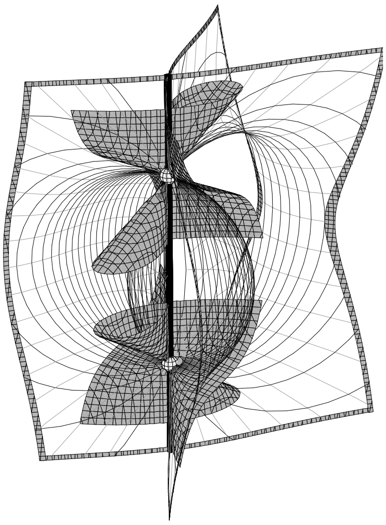

Fig. 1

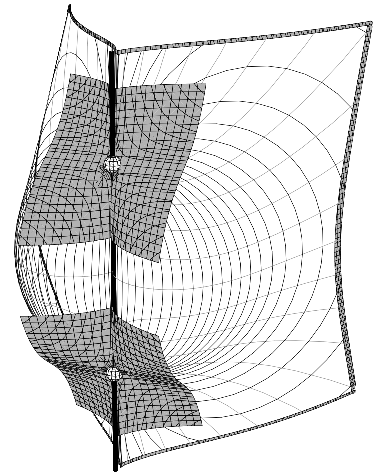

Fig. 2

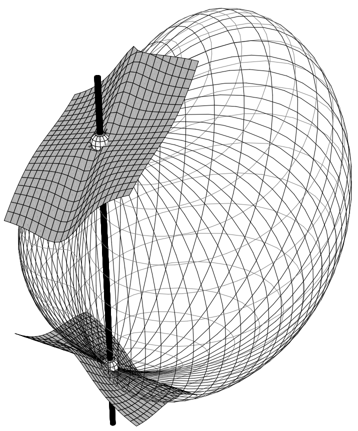

Fig. 3

The figures 1-3 show three types of the pedal surfaces of (1,2) congreunces with two triple points (types $\mathrm{VI}_1$, $\mathrm{II}_1$ and $\mathrm{I}_1$ according to [2]). The directing curves of the congruences are the double line of the surface and a hyperbola, a parabola and an ellipse. The pole lies on the double line. For the surfaces and the tangent cones at their triple points which are shown in the figures 1-3 the double line is nodal, cuspidal and isolated, respectively.

Figure 1: $(x^2 + y^2 + z^2)(x^2 - 2y^2) - 2x^3 - 2y^2x + 3(2y^2 - x^2)z = 0$

2 triple points (proper tangent cones) and ordinary binodes on a double line

Figure 2: $(x^2 + y^2 + z^2)x^2 - 2x^2y - 2y^3 - 3x^2z = 0$

2 triple points (proper tangent cones) and pinch-points (unodes) on the cuspidal line

Figure 3: $(x^2 + y^2 + z^2)(x^2 + 3y^2) - 2x^3 + 2x^2y - 2xy^2 + 2y^3 - 2(x^2 + 3y^2)z = 0$

2 triple points (proper tangent cones) and isolated binodes on a double line

The considered quartics can also contain only one triple point with a proper tangent cone. Then they can contain at least two pinch-points on the double line. The figures 4 and 5 show two examples of the pedal surfaces of (1,2) congruences (types $\mathrm{I}_{3,1}$ and $\mathrm{V}_1$ according to [2].

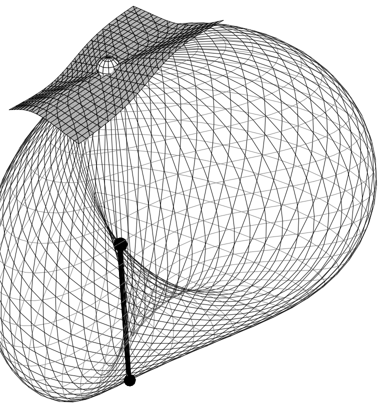
Fig. 4

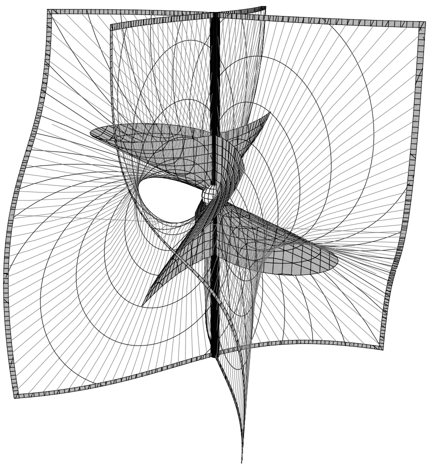
Fig. 5

Figure 4: $(x^2+y^2+z^2)(x^2+y^2)+2x^3+4x^2y+2xy^2+4y^3+2(3x^2-xy+6y^2)z=0$
1 triple point (a proper tangent cone), 2 pinch-points, ordinary and isolated binodes on a double line

Figure 5: $(x^2+y^2+z^2)xy-x^2y-y^3+(x^2+2xy-2y^2)z=0$
1 triple point (a proper tangent cone), ordinary binodes on a double line

If a tangent cone at a triple point on the considered quartics degenerates into the plane and a 2nd order cone, that cone must be real, because it contains a double line of the surface as a real ruling. It is the case 2 from the theorem 2. The figure 6 shows the pedal suface of the type $\mathrm{I}_{3,2}$ (according to [2]).

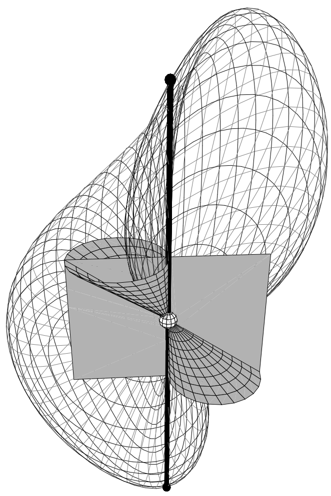
Fig. 6

Figure 6: $(x^2+y^2+z^2)(x^2+y^2)-x^3+x^2y-xy^2+y^3+(x^2-y^2)z=0$
1 triple point (a degenerated tangent cone, a 2nd order cone and a plane), 2 pinch-points, ordinary and isolated binodes on a double line

In the figures 7-9 the tangent cones at triple points degenerate into three planes. One of the tangent planes does not contain the double line of the surface. We obtained the equations of the surfaces by adding the polinomyals of the degenerated tangent cones and the polynomial $(x^2+y^2+z^2)(x^2+y^2)$. Therefore, the surfaces in the figures 7-9 contain the absolute conic and a pair of isotropic lines at infinity. These surfaces illustrate the case 3 from the theorem 2.

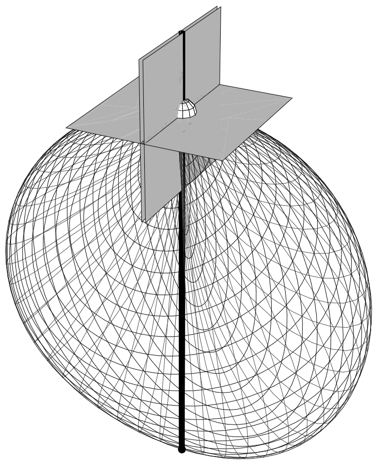

Fig. 8

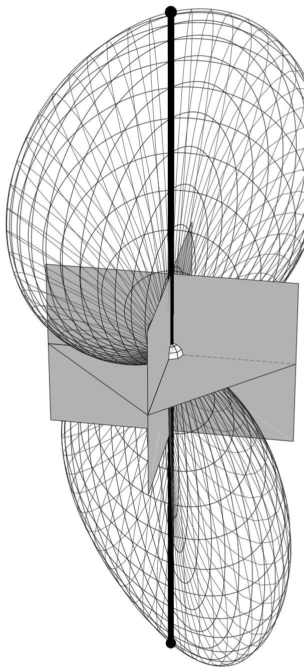

Fig. 7

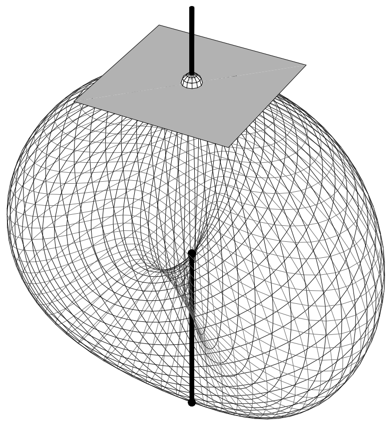

Fig. 9

Figure 7: $(x^2+y^2+z^2)(x^2+y^2)+2z(x^2-y^2)=0$

1 triple point (a degenerated tangent cone, 3 real and different non collinear planes), 2 pinch-points, ordinary and isolated binodes on a double line

Figure 8: $(x^2+y^2+z^2)(x^2+y^2)+zx^2=0$

1 triple point (a degenerated tangent cone, 3 real planes, 2 coinciding), 1 pinch-point, ordinary and isolated binodes on a double line

Figure 9: $(x^2+y^2+z^2)(x^2+y^2)+z(2x^2+y^2)=0$

1 triple point (a degenerated tangent cone, 3 non collinar planes, 2 imaginary planes through a double line), 2 pinch-points, ordinary and isolated binodes on a double line

The surfaces in the figures 10-13 contain triple points with degenerated tangent cones as in the case 4 from the theorem 2. Three tangent planes are collinear with a double line and are real and different in the example 10, two of them coincide in the example 11, two of them are imaginary in the example 12 and all the three coincide in the example 13. All other points on a double line are isolated binodes.

We obtained the equation of each surface by adding the polynomials which represent the section at infinity and the tangent cone at triple point.

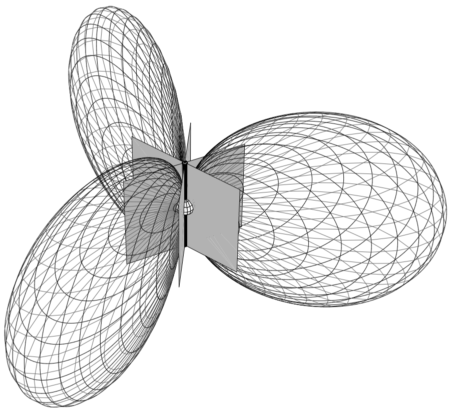

Fig.10

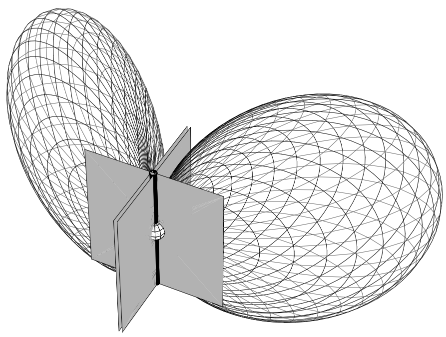

Fig.11

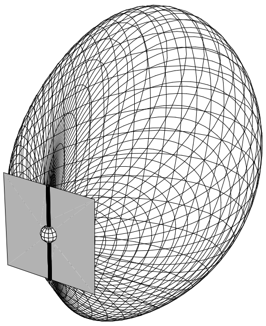

Fig.12

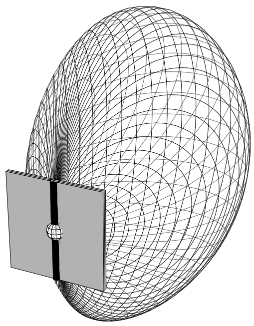

Fig. 13

Figure 10: $(x^2+y^2+z^2)(x^2+y^2)-3x^2y+y^3=0$
Figure 11: $(x^2+y^2+z^2)(x^2+y^2)+x^2y=0$
Figure 12: $(x^2+y^2+z^2)(2x^2+y^2)-2y(x^2+y^2)=0$
Figure 13: $(x^2+y^2+z^2)(x^2+y^2)+x^3=0$

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