Rose Surfaces and their Visualizations

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Abstract

In this paper we construct a new class of algebraic surfaces in three-dimensional Euclidean space that are generated by roses. We derive their parametric and implicit equations, investigate their singularities and visualize them with the program Mathematica.

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1. Introduction

In [2], by using an \((n+2)\)-degree inversion defined in [1], we elaborated the pedal surfaces of special first order line congruences. The directing lines of these congruences are roses given by the polar equation \(r = \cos n\varphi\), where \(n\) is an odd positive integer. The cases with special positions of the pole appeared to be very interesting and led us to explore a new construction of surfaces where the generating curve was a rose with a finite number of petals. The resulting surfaces had various attractive shapes, a small number of high singularities and were convenient for algebraic treatment and visualization in the program Mathematica. Another attempt to generalize roses is given in [6].

2. Roses

*Roses or rhodonea* curves \(R(n, d)\), treated here, can be expressed by the following polar equation:

\[
r = \cos \frac{n}{d} \varphi, \tag{1}
\]
where \( \frac{n}{d} \) is a positive rational number in the simplest form, i.e. \( \text{GCD}(n, d) = 1 \).

If \( n \cdot d \) is odd, the curves close at polar angles \( d \cdot \pi \) and have \( n \) petals. They are algebraic curves of the order \( n + d \), with an \( n \)-ple point in the origin and with \( \frac{1}{2} n(d - 1) \) double points. If \( n \cdot d \) is even, the curves close at polar angles \( 2d \cdot \pi \) and have \( 2n \) petals. They are algebraic curves of the order \( 2(n + d) \), with a \( 2n \)-ple point in the origin and with \( 2n(d - 1) \) double points \([5, pp. 358-369]\), \([7]\), \([8]\), \([9]\) (see Table 1).

\[
\begin{array}{|c|c|c|c|c|}
\hline
n \cdot d & \text{order} & \text{point } O & \text{number of double points} & \text{period} & \text{number of petals} \\
\hline
\text{odd} & n + d & n\text{-ple} & \frac{1}{2} n(d - 1) & d \cdot \pi & n \\
\hline
\text{even} & 2(n + d) & 2n\text{-ple} & 2n(d - 1) & 2d \cdot \pi & 2n \\
\hline
\end{array}
\]

Table 1: Properties of \( R(n, d) \)

According to \([5]\) we can derive the following implicit equation of \( R(n, d) \):

\[
\left( \sum_{k=0}^{\left\lfloor d/2 \right\rfloor} \sum_{j=0}^{k} (-1)^{k+j} \binom{d}{2k} \binom{k}{j} (x^2 + y^2)^{\frac{n+d}{2} - k+j} \right) s - \left( \sum_{i=0}^{\left\lfloor n/2 \right\rfloor} (-1)^i \binom{n}{2i} x^{n-2i} y^{2i} \right) s = 0,
\]

where \( s = 1 \) if \( n \cdot d \) is odd and \( s = 2 \) if \( n \cdot d \) is even.

According to \([4, p. 251]\)\(^{1}\), the tangent lines at the origin are given by the following equations:

- if \( n \cdot d \) is odd

\[
\sum_{k=0}^{\left\lfloor n/2 \right\rfloor} (-1)^k \binom{n}{2k} x^{n-2k} y^{2k} = 0,
\]

(3)

- if \( n \) is even

\[
\left( \sum_{k=0}^{\left\lfloor n/2 \right\rfloor} (-1)^k \binom{n}{2k} x^{n-2k} y^{2k} \right)^2 = 0,
\]

(4)

- if \( d \) is even (\( \left\lfloor d/2 \right\rfloor = \frac{d}{2} \))

\[
(x^2 + y^2)^n - \left( \sum_{k=0}^{\left\lfloor n/2 \right\rfloor} (-1)^k \binom{n}{2k} x^{n-2k} y^{2k} \right)^2 = 0.
\]

(5)

\(^{1}\)See the quotation that follows in the proof of theorem 1.
Figure 1: If $n$ is odd, the rose $R(n, 1)$ is an $n$—petalled curve with $n$ tangent lines at the origin (Figs. a and b). If $n$ is even, the rose $R(n, 1)$ is an $2n$—petalled curve with $n$ double tangent lines at the origin (Figs. c and d).

Figure 2: If $d$ is odd, the rose $R(1, d)$ has only one petal (Figs. a and b). If $d$ is even, the rose $R(1, d)$ has two petals (Figs. c and d).

Figure 3: Four roses with petals in different colors.

3. Rose Surfaces

Definition 1 Let $P(0, 0, p)$ be any point on the axis $z$ and let $R(n, d)$ be a rose given by eq. (1) in the plane $z = 0$. A rose surface $\mathcal{R}(n, d, p)$ is the system of circles $c_i$ which
lie in the planes \( \zeta \) through the axis \( z \) and have diameters \( \overline{PR_i} \), where \( R_i \neq O \) are the intersection points of the rose \( R(n, d) \) and the plane \( \zeta \) (see Fig. 4).

Figure 4: If \( n \cdot d \) is odd or even, the number of points \( R_i \in \zeta \) is \( d \) or \( 2d \), respectively.

### 3.1. Parametric equations of \( \mathcal{R}(n, d, p) \)

Let \( \varphi \) be the angle between the planes \( \zeta(\varphi) \) and \( y = 0 \). The parametric equations of the circle \( c \) with the diameter \( \overline{PR} \) in the plane \( \zeta(\varphi) \) are the following:

\[
\begin{align*}
r &= \frac{1}{2} \cos \frac{n}{d} \varphi + \sqrt{p^2 + \cos^2 \frac{n}{d} \varphi \sin \theta} \\
z &= \frac{1}{2} (p + \sqrt{p^2 + \cos^2 \frac{n}{d} \varphi \cos \theta}), \quad \theta \in [0, 2\pi),
\end{align*}
\]

where \( \varphi \in [0, d \cdot \pi) \) if \( n \cdot d \) is odd, and \( \varphi \in [0, 2d \cdot \pi) \) if \( n \cdot d \) is even.

Therefore, the parametric equations of the rose surface \( \mathcal{R}(n, d, p) \) are the following:

\[
\begin{align*}
x &= \frac{1}{2} \cos \varphi (\cos \frac{n}{d} \varphi + \sqrt{p^2 + \cos^2 \frac{n}{d} \varphi \sin \theta}) \\
y &= \frac{1}{2} \sin \varphi (\cos \frac{n}{d} \varphi + \sqrt{p^2 + \cos^2 \frac{n}{d} \varphi \sin \theta}) \\
z &= \frac{1}{2} (p + \sqrt{p^2 + \cos^2 \frac{n}{d} \varphi \cos \theta}),
\end{align*}
\]
where \((\varphi, \theta) \in [0, d \cdot \pi) \times [0, 2\pi)\) if \(n \cdot d\) is odd, and \((\varphi, \theta) \in [0, 2d \cdot \pi) \times [0, 2\pi)\) if \(n \cdot d\) is even.

Equations (7) allow for Mathematica visualizations of surfaces \(\mathcal{R}(n, d, p)\), see [3].

### 3.2. Implicit equations of \(\mathcal{R}(n, d, p)\)

From the identity \(\cos \frac{d}{2} \varphi = \cos n \varphi\) and multiple angle formula

\[
\cos n \varphi = \sum_{i=0}^{[n/2]} (-1)^i \binom{n}{2i} (\sin \varphi)^{2i} (\cos \varphi)^{n-2i}
\]

we obtain

\[
\sum_{k=0}^{[d/2]} (-1)^k \binom{d}{2k} \left( \sin \frac{n}{d} \varphi \right)^{2k} (\cos \frac{n}{d} \varphi)^{d-2k} = \sum_{i=0}^{[n/2]} (-1)^i \binom{n}{2i} (\sin \varphi)^{2i} (\cos \varphi)^{n-2i}. \quad (8)
\]

Since the implicit equation of the circle \(c\) in the plane \(\zeta(\varphi)\) is

\[
\left( r - \frac{\cos \frac{n}{d} \varphi}{2} \right)^2 + \left( z - \frac{p}{2} \right)^2 = \frac{1}{4} \left( \cos^2 \frac{n}{d} \varphi + p^2 \right), \quad (9)
\]

by substituting \(r = \sqrt{x^2 + y^2}\) in (9), we obtain the following conditions for the points of \(\mathcal{R}(n, d, p)\):

\[
\cos \frac{n}{d} \varphi = \frac{x^2 + y^2 + z^2 - p \cdot z}{\sqrt{x^2 + y^2}}, \quad \sin \frac{n}{d} \varphi = \sqrt{1 - \frac{(x^2 + y^2 + z^2 - p \cdot z)^2}{x^2 + y^2}}. \quad (10)
\]

By substituting (10) and \(\cos \varphi = \frac{x}{\sqrt{x^2+y^2}}, \sin \varphi = \frac{y}{\sqrt{x^2+y^2}}\) into equation (8), we obtain the following algebraic equations:

\[
(x^2 + y^2)^{s(d-n)} \left( \sum_{k=0}^{[d/2]} \binom{d}{2k} \left( \frac{n}{2d} \right)^{k-j} \left( \frac{n}{2d} \right)^{j-k} \right)^s = \sum_{i=0}^{[n/2]} (-1)^i \binom{n}{2i} x^{n-2i} y^{2i}.
\]

\[
(x^2 + y^2)^{s(d-n)} \left( \sum_{k=0}^{[d/2]} \binom{d}{2k} \left( \frac{n}{2d} \right)^{k-j} \left( \frac{n}{2d} \right)^{j-k} \right)^s = \sum_{i=0}^{[n/2]} (-1)^i \binom{n}{2i} x^{n-2i} y^{2i}.
\]

\[
(x^2 + y^2)^{s(d-n)} \left( \sum_{k=0}^{[d/2]} \binom{d}{2k} \left( \frac{n}{2d} \right)^{k-j} \left( \frac{n}{2d} \right)^{j-k} \right)^s = \sum_{i=0}^{[n/2]} (-1)^i \binom{n}{2i} x^{n-2i} y^{2i}.
\]
where \( s = 1 \) if \( n \cdot d \) is odd, and \( s = 2 \) if \( n \cdot d \) is even.

Equations (11) and (12) are the implicit equations of \( R(n, d, p) \) for \( n > d \) and \( n < d \), respectively.

### 3.3. Properties of \( R(n, d, p) \)

**Theorem 1** For rose surfaces \( R(n, d, p) \), the following table is valid:

\[
\begin{array}{|c|c|c|c|c|}
\hline
n \cdot d & \text{order} & \text{points } O \text{ and } P & \text{axis } z & \text{double circles in } \zeta \\
\hline
\text{odd} & n > d & n + d & n\text{-ple} & (n - d)\text{-ple} & \frac{1}{2}n(d - 1) \\
\hline
\text{even} & n > d & 2(n + d) & 2n\text{-ple} & 2(n - d)\text{-ple} & n(2d - 1) \\
\hline
\text{odd} & n < d & 2d & d\text{-ple} & 0\text{-ple} & \frac{1}{2}n(d - 1) \\
\hline
\text{even} & n < d & 4d & 2d\text{-ple} & 0\text{-ple} & n(2d - 1) \\
\hline
\end{array}
\]

**Table 2**: Properties of \( R(n, d, p) \).

**Proof:**

**ad A** The order of an algebraic surface is equal to the degree of its algebraic equation. In eqs. (11) and (12) the terms with the highest exponents (for \( k = j \)) are

\[
\left(2^{d-1}(x^2 + y^2)^{\frac{n-d}{2}}(x^2 + y^2 + z^2)^d\right)^s
\]

and

\[
\left(2^{d-1}(x^2 + y^2 + z^2)^d\right)^s - (x^2 + y^2)^{\frac{d(n-a)}{2}} \left( \sum_{i=0}^{\lfloor \frac{a}{2} \rfloor} (-1)^i \binom{n}{2i} x^{n-2i} y^{2i} \right)^s,
\]

**ad B** According to [4, p.251]: If an \( n \)th order surface in \( \mathbb{E}^3 \) which passes through the origin is given by the equation

\[
F(x, z, y) = f_m(x, y, z) + f_{m+1}(x, y, z) + \cdots + f_n(x, y, z) = 0,
\]
where \( f_k(x, y, z) (1 \leq k \leq n) \) are homogeneous polynomials of degree \( k \), then the tangent cone at the origin is given by the equation \( f_n(x, y, z) = 0 \).

Therefore, the tangent cones of \( \mathcal{R}(n, d, p) \) given by eqs. (11) and (12) at their points \( O \) and \( P \) are given by the following equations:

\[
\left( \sum_{i=0}^{\left\lfloor d^2/2 \right\rfloor} (-1)^i \binom{n}{2i} x^{n-2i} y^{2i} (x^2 + y^2)^{\frac{d-n}{2}} \right)^s = 0, \
\left( \sum_{k=0}^{\left\lfloor n^2/2 \right\rfloor} (-1)^k \binom{n}{2k} (x^2 + y^2)^{\frac{n-d+2(k-j)}{2}} \right)^s = 0,
\]

respectively.

In these equations, \(-p \cdot d\) corresponds with the point \( O \), and \(+p \cdot d\) with the point \( P \) as the origin.

**Ad C** If \( \mathcal{R}(n, d, p) \) is given by eq. (11), any point \( Z_0(0, 0, z_0) \) lies on the surface and the tangent cone at \( Z_0 \), with the origin translated into \( Z_0 \), is given by the following equation:

\[
(x^2 + y^2)^{\frac{a(n-d)}{2}} = 0.
\]

This equation represents the \( \frac{a(n-d)}{2} \)-ple pair of isotropic planes through the axis \( z \).

If \( \mathcal{R}(n, d, p) \) is given by eq. (12), it is clear that a point \( Z_0(0, 0, z_0) \) on the axis \( z \) lies on \( \mathcal{R}(n, d, p) \) iff \( z_0^2 - p \cdot z_0 = 0 \), i.e. \( Z_0 = O \) or \( Z_0 = P \).

**Ad D** The circle \( c \) in the plane \( \zeta \) is the double curve of \( \mathcal{R}(n, d, p) \) iff the intersection point of \( \zeta \) and \( R(n, d) \) is the double point of \( R(n, d) \). Thus, the number of double circles on \( \mathcal{R}(n, d, p) \) is equal to the number of double points of \( R(n, d) \) if \( n \cdot d \) is odd. But, if \( n \cdot d \) is even, other \( n \) double circles in the planes \( \zeta \) exist on \( \mathcal{R}(n, d, p) \). These circles lie in the planes through the double tangent lines of \( R(n, d) \) at \( O \) and their diameters are \( OP \). If \( O = P \), these circles degenerate into the pairs of isotropic lines. \( \square \)
**Corollary 1** If $p = 0$, the tangent cone of $\mathcal{R}(n, d, p)$ at $O = P$ splits into $n$ or $d$ planes.

**Proof:** If $p = 0$, eqs. (13) and (14) take the following forms:

\[
\left( \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n}{2i} x^{n-2i} y^{2i} \right)^s = 0, \tag{16}
\]

\[
\left( \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n}{2i} x^{n-2i} y^{2i} (x^2 + y^2)^{\frac{d}{2i}} \right)^s = 0, \tag{17}
\]

respectively.

Since the polynomials in these equations are $n$th (eq. 16) or $d$th (eq. 17) degree homogeneous in $x$ and $y$, therefore they can be reduced to linear and quadratic factors. These factors equal to 0 represent $n$ or $d$ planes (real or imaginary) through the axis $z$. 

\[\square\]

**3.4. Visualizations of $\mathcal{R}(n, d, p)$**

The following figures are computed and plotted by the software *Mathematica*.

Figure 5: If $d = 1$ and $p \neq 0$, the tangent cones at the points $O$ and $P$ are proper cones. If $n$ is odd, there are no double circles on $\mathcal{R}(n, 1, p)$. If $n$ is even, $2n$ double circles exist on $\mathcal{R}(n, 1, p)$.
Figure 6: If $d = 1$ and $p = 0$, the tangent cones at $O$ split into $n$ planes. If $n$ is even, these planes are the double tangent planes of $\mathcal{R}(n, 1, 0)$.

Figure 7: Four rose surfaces.

**References**


Figure 8: Seven rose surfaces.


[6] "Twisting Rose Surfaces” from The Wolfram Demonstrations Project
   http://demonstrations.wolfram.com/TwistingRoseSurfaces/
   Contributed by: George Back

   http://mathworld.wolfram.com/Rose.html
