Rose Surfaces and their Visualizations

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Abstract

In this paper we construct a new class of algebraic surfaces in three-dimensional Euclidean space that are generated by roses. We derive their parametric and implicit equations, investigate their singularities and visualize them with the program *Mathematica*.

Keywords: Rose curve, Rose surface, Singular point
MSC 2000: 51N20, 51M15, 14J25, 14J17

1. Introduction

In [2], by using an $(n+2)$-degree inversion defined in [1], we elaborated the pedal surfaces of special first order line congruences. The directing lines of these congruences are roses given by the polar equation $r = \cos n\varphi$, where n is an odd positive integer. The cases with special positions of the pole appeared to be very interesting and led us to explore a new construction of surfaces where the generating curve was a rose with a finite number of petals. The resulting surfaces had various attractive shapes, a small number of high singularities and were convenient for algebraic treatment and visualization in the program *Mathematica*. Another attempt to generalize roses is given in [6].

2. Roses

Roses or *rhodonea* curves $R(n, d)$, treated here, can be expressed by the following polar equation:

$$r = \cos \frac{n}{d}\varphi, \tag{1}$$

where $\frac{n}{d}$ is a positive rational number in the simplest form, i.e. $GCD(n, d) = 1$.

If $n \cdot d$ is *odd*, the curves close at polar angles $d \cdot \pi$ and have n petals. They are algebraic curves of the order $n + d$, with an n-ple point in the origin and with $\frac{1}{2}n(d-1)$ double points. If $n \cdot d$ is *even*, the curves close at polar angles $2d \cdot \pi$ and have $2n$ petals. They are algebraic curves of the order $2(n+d)$, with a $2n$-ple point in the origin and with $2n(d-1)$ double points [5, pp. 358-369], [7], [8], [9] (see Table 1).

$n \cdot d$	order	point O	number of double points	period	number of petals
odd	$n + d$	n-ple	$\frac{1}{2}n(d-1)$	$d \cdot \pi$	n
even	$2(n+d)$	$2n$-ple	$2n(d-1)$	$2d \cdot \pi$	$2n$

Table 1: Properties of $R(n, d)$

According to [5] we can derive the following implicit equation of $R(n, d)$:

$$\Big(\sum_{k=0}^{\lfloor d/2 \rfloor} \sum_{j=0}^{k} (-1)^{k+j} \binom{d}{2k} \binom{k}{j} (x^2+y^2)^{\frac{n+d}{2}-k+j}\Big)^s - \Big(\sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \binom{n}{2i} x^{n-2i} y^{2i}\Big)^s = 0, \quad (2)$$

where $s = 1$ if $n \cdot d$ is odd and $s = 2$ if $n \cdot d$ is even.

According to [4, p. 251][1], the tangent lines at the origin are given by the following equations:

- if $n \cdot d$ is odd

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} x^{n-2k} y^{2k} = 0, \quad (3)$$

- if n is even

$$\Big(\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} x^{n-2k} y^{2k}\Big)^2 = 0, \quad (4)$$

- if d is even ($\lfloor \frac{d}{2} \rfloor = \frac{d}{2}$)

$$(x^2+y^2)^n - \Big(\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} x^{n-2k} y^{2k}\Big)^2 = 0. \quad (5)$$

[1] See the quotation that follows in the proof of theorem 1.

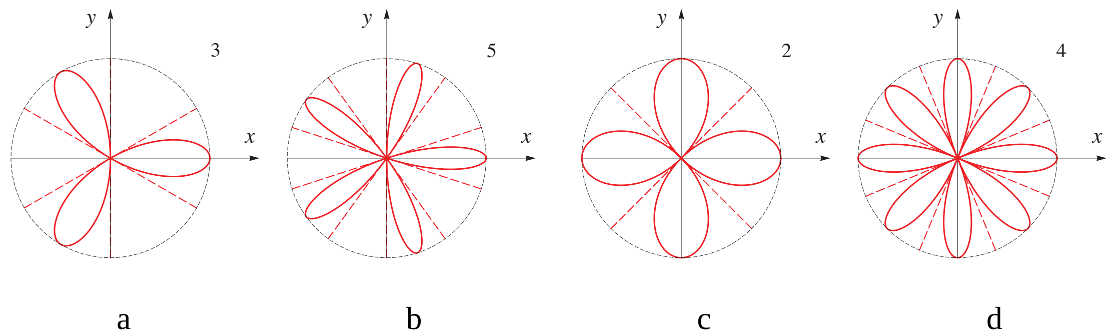


Figure 1: If n is odd, the rose $R(n, 1)$ is an $n-$petalled curve with n tangent lines at the origin (Figs. a and b). If n is even, the rose $R(n, 1)$ is an $2n-$petalled curve with n double tangent lines at the origin (Figs. c and d).

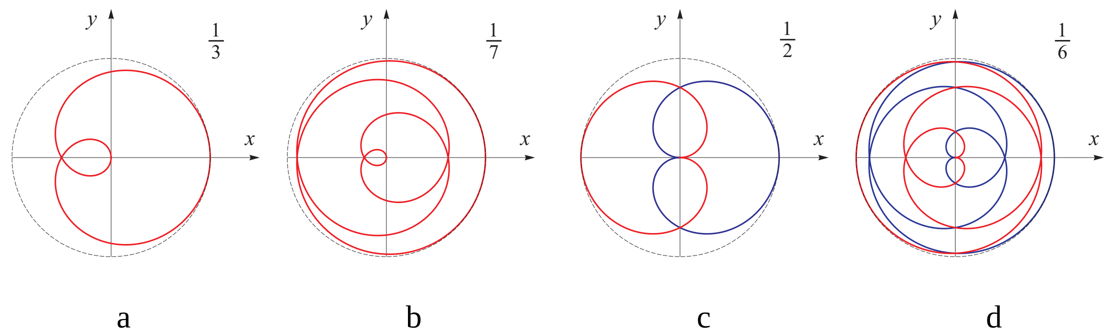


Figure 2: If d is odd, the rose $R(1, d)$ has only one petal (Figs. a and b). If d is even, the rose $R(1, d)$ has two petals (Figs. c and d).

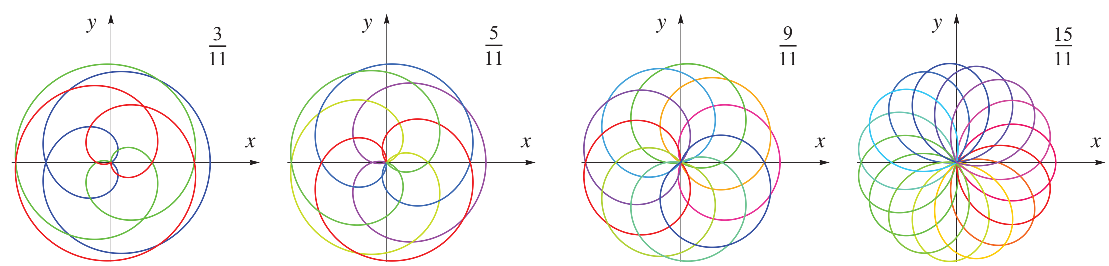


Figure 3: Four roses with petals in different colors.

3. Rose Surfaces

Definition 1 *Let* $P(0, 0, p)$ *be any point on the axis* z *and let* $R(n, d)$ *be a rose given by eq. (1) in the plane* $z = 0$*. A rose surface* $\mathcal{R}(n, d, p)$ *is the system of circles* c_i *which* *lie in the planes ζ through the axis z and have diameters $\overline{PR_i}$, where $R_i \neq O$ are the intersection points of the rose $R(n, d)$ and the plane ζ (see Fig. 4).*

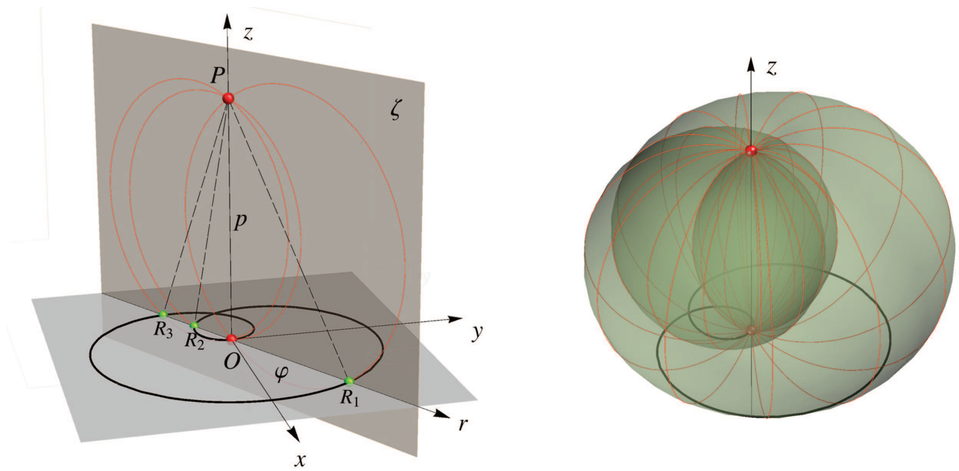


Figure 4: If $n \cdot d$ is odd or even, the number of points $R_i \in \zeta$ is d or $2d$, respectively.

3.1. Parametric equations of $\mathcal{R}(n, d, p)$

Let φ be the angle between the planes $\zeta(\varphi)$ and $y = 0$. The parametric equations of the circle c with the diameter $\overline{PR}$ in the plane $\zeta(\varphi)$ are the following:

$$
\begin{aligned}
r &= \frac{1}{2}\cos\frac{n}{d}\varphi + \sqrt{p^2 + \cos^2\frac{n}{d}\varphi}\sin\theta) \\
z &= \frac{1}{2}(p + \sqrt{p^2 + \cos^2\frac{n}{d}\varphi}\cos\theta), \quad \theta \in [0, 2\pi),
\end{aligned} \tag{6}
$$

where $\varphi \in [0, d \cdot \pi)$ if $n \cdot d$ is odd, and $\varphi \in [0, 2d \cdot \pi)$ if $n \cdot d$ is even.
Therefore, the parametric equations of the rose surface $\mathcal{R}(n, d, p)$ are the following:

$$
\begin{aligned}
x &= \frac{1}{2}\cos\varphi(\cos\frac{n}{d}\varphi + \sqrt{p^2 + \cos^2\frac{n}{d}\varphi}\sin\theta) \\
y &= \frac{1}{2}\sin\varphi(\cos\frac{n}{d}\varphi + \sqrt{p^2 + \cos^2\frac{n}{d}\varphi}\sin\theta) \\
z &= \frac{1}{2}(p + \sqrt{p^2 + \cos^2\frac{n}{d}\varphi}\cos\theta),
\end{aligned} \tag{7}
$$

where $(\varphi, \theta) \in [0, d \cdot \pi) \times [0, 2\pi)$ if $n \cdot d$ is odd, and $(\varphi, \theta)) \in [0, 2d \cdot \pi) \times [0, 2\pi)$ if $n \cdot d$ is even.

Equations (7) allow for *Mathematica* visualizations of surfaces $\mathcal{R}(n, d, p)$, see [3].

3.2. Implicit equations of $\mathcal{R}(n, d, p)$

From the identity $\cos d\frac{n}{d}\varphi = \cos n\varphi$ and multiple angle formula $\cos n\varphi = \sum_{i=0}^{\lfloor n/2 \rfloor}(-1)^i\binom{n}{2i}(\sin\varphi)^{2i}(\cos\varphi)^{n-2i}$ we obtain

$$\sum_{k=0}^{\lfloor d/2 \rfloor}(-1)^k\binom{d}{2k}(\sin\frac{n}{d}\varphi)^{2k}(\cos\frac{n}{d}\varphi)^{d-2k} = \sum_{i=0}^{\lfloor n/2 \rfloor}(-1)^i\binom{n}{2i}(\sin\varphi)^{2i}(\cos\varphi)^{n-2i}. \quad (8)$$

Since the implicite equation of the circle c in the plane $\zeta(\varphi)$ is

$$\left(r - \frac{\cos\frac{n}{d}\varphi}{2}\right)^2 + \left(z - \frac{p}{2}\right)^2 = \frac{1}{4}\,(\cos^2\frac{n}{d}\varphi + p^2), \quad (9)$$

by substituting $r = \sqrt{x^2 + y^2}$ in (9), we obtain the following conditions for the points of $\mathcal{R}(n, d, p)$:

$$\cos\frac{n}{d}\varphi = \frac{x^2 + y^2 + z^2 - p \cdot z}{\sqrt{x^2 + y^2}}, \quad \sin\frac{n}{d}\varphi = \sqrt{1 - \frac{(x^2 + y^2 + z^2 - p \cdot z)^2}{x^2 + y^2}}. \quad (10)$$

By substituting (10) and $\cos\varphi = \frac{x}{\sqrt{x^2+y^2}}$, $\sin\varphi = \frac{y}{\sqrt{x^2+y^2}}$ into equation (8), we obtain the following algebraic equations:

$$(x^2 + y^2)^{\frac{s(n-d)}{2}} \left(\sum_{k=0}^{\lfloor \frac{d}{2} \rfloor}\sum_{j=0}^{k}(-1)^{k+j}\binom{d}{2k}\binom{k}{j}(x^2 + y^2 + z^2 - p \cdot z)^{d-2(k-j)}(x^2 + y^2)^{k-j}\right)^s$$
$$= \left(\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^i\binom{n}{2i}x^{n-2i}y^{2i}\right)^s, \quad (11)$$

$$\left(\sum_{k=0}^{\lfloor \frac{d}{2} \rfloor}\sum_{j=0}^{k}(-1)^{k+j}\binom{d}{2k}\binom{k}{j}(x^2 + y^2 + z^2 - p \cdot z)^{d-2(k-j)}(x^2 + y^2)^{k-j}\right)^s$$
$$=(x^2 + y^2)^{\frac{s(d-n)}{2}} \left(\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^i\binom{n}{2i}x^{n-2i}y^{2i}\right)^s, \quad (12)$$

where $s = 1$ if $n \cdot d$ is odd, and $s = 2$ if $n \cdot d$ is even.

Equations (11) and (12) are the implicit equations of $\mathcal{R}(n, d, p)$ for $n > d$ and $n < d$, respectively.

3.3. Properties of $\mathcal{R}(n, d, p)$

Theorem 1 *For rose surfaces $\mathcal{R}(n, d, p)$, the following table is valid:*

$n \cdot d$		order	points O and P	axis z	double circles in ζ
odd	$n > d$	$n + d$	n-ple	$(n - d)$-ple	$\frac{1}{2}n(d - 1)$
even	$n > d$	$2(n + d)$	$2n$-ple	$2(n - d)$-ple	$n(2d - 1)$
odd	$n < d$	$2d$	d-ple	0-ple	$\frac{1}{2}n(d - 1)$
even	$n < d$	$4d$	$2d$-ple	0-ple	$n(2d - 1)$
		A	**B**	**C**	**D**

Table 2: Properties of $\mathcal{R}(n, d, p)$.

PROOF:

ad **A** The order of an algebraic surface is equal to the degree of its algebraic equation. In eqs. (11) and (12) the terms with the highest exponents (for $k = j$) are
$\left(2^{d-1}(x^2 + y^2)^{\frac{n-d}{2}}(x^2 + y^2 + z^2)^d\right)^s$ and
$\left(2^{d-1}(x^2 + y^2 + z^2)^d\right)^s - (x^2 + y^2)^{\frac{s(d-n)}{2}}\left(\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^i\binom{n}{2i}x^{n-2i}y^{2i}\right)^s$, respectively.

ad **B** According to [4, p.251]: If an nth order surface in $\mathbb{E}^3$ which passes through the origin is given by the equation

$$F(x, z, y) = f_m(x, y, z) + f_{m+1}(x, y, z) + \cdots + f_n(x, y, z) = 0,$$

where $f_k(x, y, z)$ ($1 \leq k \leq n$) are homogeneous polynomials of degree k, then the tangent cone at the origin is given by the equation $f_m(x, y, z) = 0$.

Therefore, the tangent cones of $\mathcal{R}(n, d, p)$ given by eqs. (11) and (12) at their points O and P are given by the following equations:

$$\left(\sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} \sum_{j=0}^{k} (-1)^{k+j} \binom{d}{2k} \binom{k}{j} (\mp p \cdot z)^{d-2(k-j)} (x^2 + y^2)^{\frac{n-d+2(k-j)}{2}} \right)^s - \left(\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n}{2i} x^{n-2i} y^{2i} \right)^s = 0, \tag{13}$$

$$\left(\sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} \sum_{j=0}^{k} (-1)^{k+j} \binom{d}{2k} \binom{k}{j} (\mp p \cdot z)^{d-2(k-j)} (x^2 + y^2)^{k-j} \right)^s - \left(\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n}{2i} x^{n-2i} y^{2i} (x^2 + y^2)^{\frac{d-n}{2}} \right)^s = 0, \tag{14}$$

respectively.

In these equations, $-p \cdot d$ corresponds with the point O, and $+p \cdot d$ with the point P as the origin.

ad **C** If $\mathcal{R}(n, d, p)$ is given by eq. (11), any point $Z_0(0, 0, z_0)$ lies on the surface and the tangent cone at Z_0, with the origin translated into Z_0, is given by the following equation:

$$(x^2 + y^2)^{\frac{s(n-d)}{2}} = 0. \tag{15}$$

This equation represents the $\frac{s(n-d)}{2}$-ple pair of isotropic planes through the axis z.

If $\mathcal{R}(n, d, p)$ is given by eq. (12), it is clear that a point $Z_0(0, 0, z_0)$ on the axis z lies on $\mathcal{R}(n, d, p)$ iff $z_0^2 - p \cdot z_0 = 0$, i.e. $Z_0 = O$ or $Z_0 = P$.

Ad **D** The circle c in the plane ζ is the double curve of $\mathcal{R}(n, d, p)$ iff the intersection point of ζ and $R(n, d)$ is the double point of $R(n, d)$. Thus, the number of double circles on $\mathcal{R}(n, d, p)$ is equal to the number of double points of $R(n, d)$ if $n \cdot d$ is odd. But, if $n \cdot d$ is even, other n double circles in the planes ζ exist on $\mathcal{R}(n, d, p)$. These circles lie in the planes through the double tangent lines of $R(n, d)$ at O and their diameters are $\overline{OP}$. If $O = P$, these circles degenerate into the pairs of isotropic lines. $\square$

Corollary 1 *If $p = 0$, the tangent cone of $\mathcal{R}(n, d, p)$ at $O = P$ splits into n or d planes.*

PROOF: If $p = 0$, eqs. (13) and (14) take the following forms:

$$\left(\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n}{2i} x^{n-2i} y^{2i}\right)^s = 0, \tag{16}$$

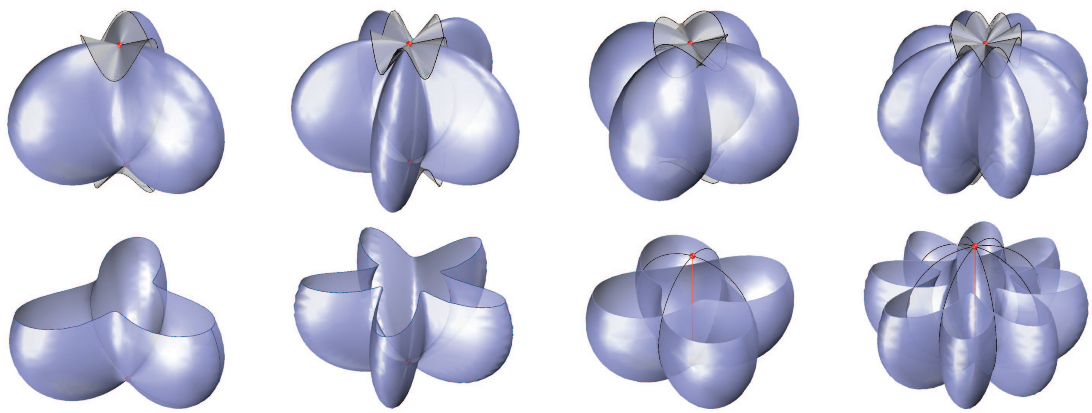
$$\left(\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n}{2i} x^{n-2i} y^{2i} (x^2 + y^2)^{\frac{d-n}{2}}\right)^s = 0, \tag{17}$$

respectively.

Since the polynomials in these equations are nth (eq. 16) or dth (eq. 17) degree homogeneous in x and y, therefore they can be reduced to linear and quadratic factors. These factors equal to 0 represent n or d planes (real or imaginary) through the axis z. □

3.4. Visualizations of $\mathcal{R}(n, d, p)$

The following figures are computed and plotted by the software *Mathematica*.



$\mathcal{R}$(3,1,2) $\mathcal{R}$(5,1,2) $\mathcal{R}$(2,1,2) $\mathcal{R}$(4,1,2)

Figure 5: If $d = 1$ and $p \neq 0$, the tangent cones at the points O and P are proper cones. If n is odd, there are no double circles on $\mathcal{R}(n, 1, p)$. If n is even, $2n$ double circles exist on $\mathcal{R}(n, 1, p)$.

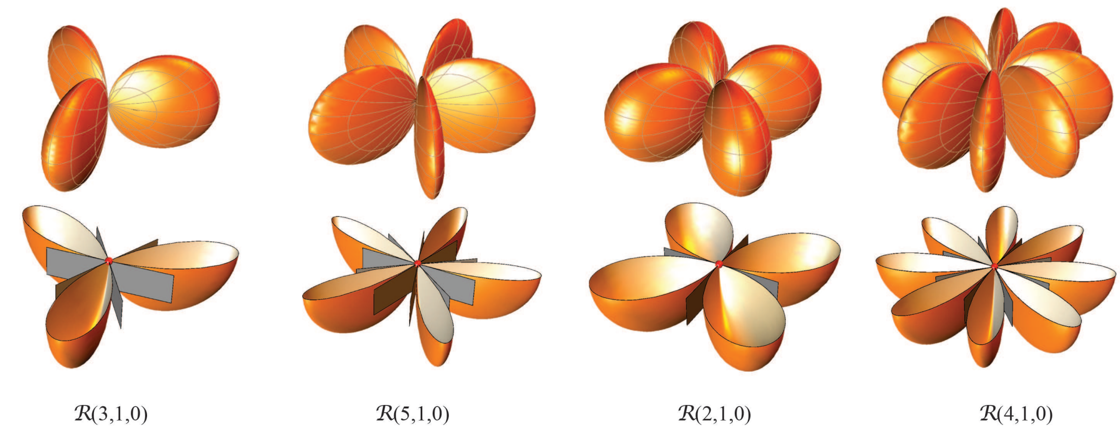


Figure 6: If $d = 1$ and $p = 0$, the tangent cones at O split into n planes. If n is even, these planes are the double tangent planes of $\mathcal{R}(n, 1, 0)$.

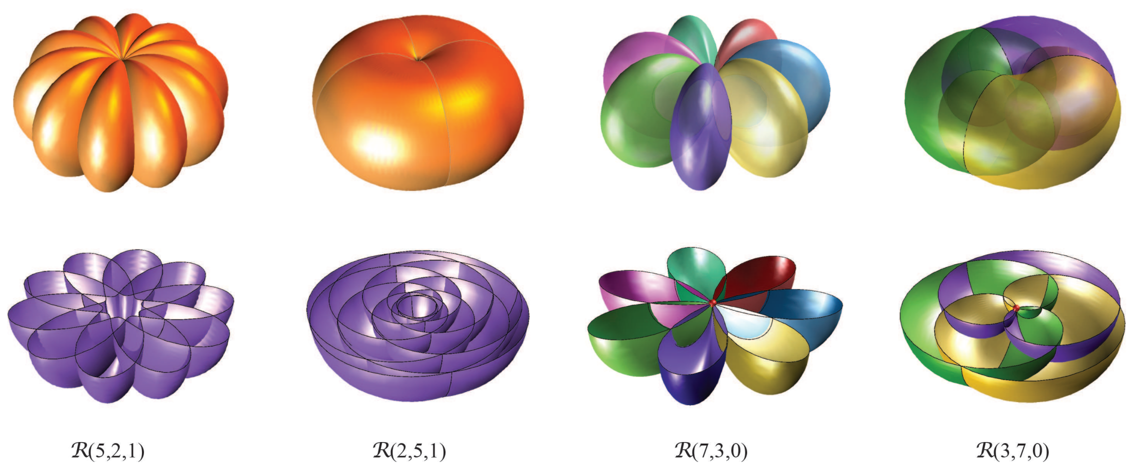


Figure 7: Four rose surfaces.

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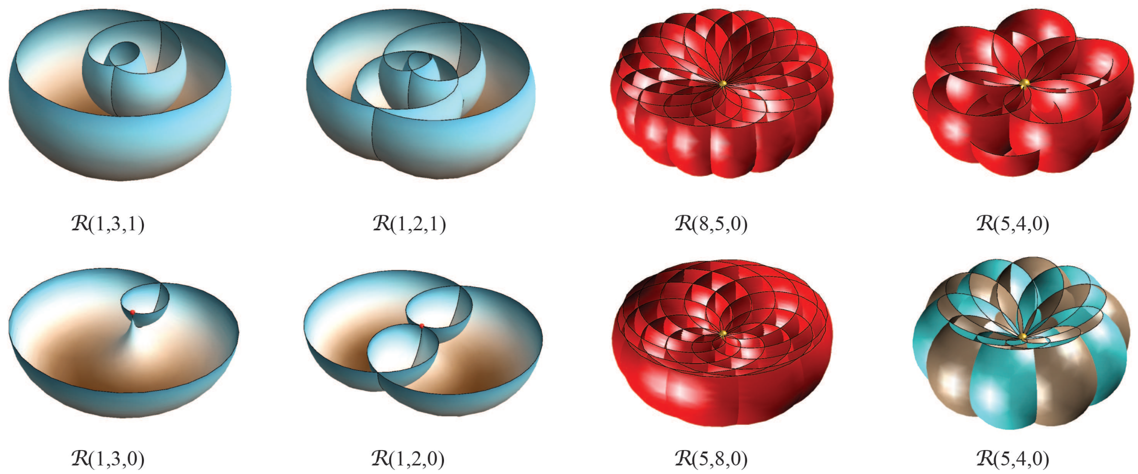

Figure 8: Seven rose surfaces.

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