Three-bar curves, asymmetric propellers and idempotent medial quasigroups

Vladimir Volenec
Faculty of Science, University of Zagreb, Zagreb, Croatia
e-mail: volenec@math.hr

The three-bar curve (invented by James Watt) is generated by a linkwork of three jointed rods va and wb, the two legs each having a pivot point at fixed points v and w. The tracting point p is attached to the traversing bar ab by a rigid triangular plate abp. The resulting three-bar curve is a sextic with triple points at two imaginary absolute points of Euclidean plane and with three (real or imaginary) double points.

S. Roberts (1876) made the discovery that the same three-bar curve can be generated by three different linkages of the same type with directly similar triangular plates. The basis for this result is the following theorem.

Theorem 1 *If pab, dpc, efp are directly similar triangles and if $dpeu$, $fpav$, $bpcw$ are parallelograms, then the triangle uvw is directly similar to the given triangles.*

In the triple geberation of S. Roberts, the pivot points u, v, w are fixed. So, there are three linkages $uefv$, $udcw$, $vabw$ with pivot points u, v; u, w; v, w and with triangular plates abp, cdp, efp, respectively.

L. Bankoff, P. Erdös and M. S. Klamkin (1973) proved the generalized asymmetric propeller theorem with the statement:

If pqr, pab, dqc, efr are directly similar triangles, then u', v', w', the midpoints of de, fa, bc, are the vertices of a triangle $u'v'w'$ directly similar to the first four triangles.

In Euclidean plane this statement is equivalent to the statement of the following theorem.

Theorem 2 *If pab, dqc, efr are directly similar triangles and if u', v', w', the midpoints of de, fa, bc, are the vertices of a triangle $u'v'w'$ directly similar to these three triangles, then all four triangles are directly similar to the triangle pqr.*

If $p = q = r$ in Theorem 2, and if the points u', v', w' are the midpoints of the segments pu, pv, pw in Theorem 1, then it is obvious that Theorem 2 implies Theorem1.

If the triangles pqr, pab, dqc, efr are equilateral, then we have the classical case of asymmetric propeller.

Here we shall study the algebraic background of Theorem 1 and Theorem 1. This background is an idempotent medial quasigroup, i.e. a quasigroup $(Q, \cdot)$ with the identities $aa = a$ and $ab \cdot cd = ac \cdot bd$.

Key words: three-bar curves, idempotent medial quasigroup, asymmetric propeller

MSC 2010: 14H50, 51N20, 20N05

References

[1] L. Bankoff, P. Erdös, M. S. Klamkin, The asymmetric propeller, *Mathematics Magazine* **46** (5) (1973), 270–272.

[2] M. Gardner, The asymmetric propeller, *The College Mathematics Journal* **30** (1999), 18–22.

[3] F. V. Morley, The three-bar curve, *The American Mathematical Monthly* **31** (1924), 71–79.

[4] S. Roberts, On three-bar motion in plane space, *Proceedings of the London Mathematical Society* **7** (1) (1876), 14–23.

[5] W. Schmid, Zur Synthese des Dreistabgetriebes, *Universidad Nacional de Tucumán. Facultad de Ciencias Exactas y Tecnología. Revista. Serie A.* **13** (1960), 158–166.

[6] V. Volenec, Geometry of medial quasigroups, *Rad Hrvatske Akademije Znanosti i Umjetnosti. Matematičke znanosti* [421] **5** (1986), 79–91.

[7] V. Volenec, Geometry of IM-quasigroups, *Rad Hrvatske Akademije Znanosti i Umjetnosti. Matematičke znanosti* [456] **10** (1991), 139–146.