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ABSTRACTS

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Plenary lectures

Learning and teaching mathematics in the era of massification of higher education

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Universities under conditions of massification are expected to meet requirements of diverse group of students, changing needs of employers, state, labor market and society as well as to fight competition in globalized educational market [2]. Students needs are resulting from differentiated student populations in the situation when universities move from an elite basis to a mass system of education. At the same time, we witness a considerable waste of resources and talents. For example, in Croatia only 10-15% of students graduate on time and annual graduation rates have never reached 50%.

Along with this, mathematics is generally viewed as a "hard" subject and obstacle for access and retention and teachers of mathematics are perceived as not ready and not trained for new challenges. One of the prevailing strategies is the avoidance of mathematics by study programs designers, by decision makers when setting enrollment criteria and finally by students whenever they can. On the other hand, teachers of mathematics are often insisting on traditional well established way in choosing mathematical content to teach in non-mathematical studies, teaching and learning methods, not distinguishing mathematical approaches and goals for different study programs and finally blaming students not to study hard enough.

Therefore, it is essential to share examples of good practice among teachers of mathematics in higher education. Even if there is no single universal formula, there are many teaching methods for teaching and learning mathematics available that incorporate activities that encourage student participation and change student attitudes towards mathematics. In many cases it means to incorporate real-world problems students could relate to and to see the value of mathematics in their personal experiences. Further, thoughtfully applied technology supported teaching and learning of mathematics [1] enhances learning mathematics, and cooperative learning of mathematics in problem-based approach stimulates deeper mathematical understanding. Finally, to help students to learn mathematics, institutions of higher education should establish Mathematics Learning and Support Centers, introduce bridging courses and peer-to-peer support.

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Kinematics and algebraic geometry

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Various mathematical formulations are used to formulate mechanism and robot kinematics. The mathematical formulation is the basis for kinematic analysis and synthesis, i.e., determining displacements, velocities and accelerations, on the one hand, and obtaining design parameters on the other. Vector/matrix formulation containing trigonometric functions is arguably the most favoured approach used in the engineering research community. A less well known but nevertheless very successful approach relies on algebraic formulation. This involves describing mechanism constraints with algebraic (polynomial) equations and solving the equation sets, that pertain to some given mechanism or robot, with the powerful tools of algebraic and *numerical* algebraic geometry.

For 15 years, now, the author and his collaborators have been applying algebraic formulation to kinematics in particular instances but wide range of analysis and synthesis problems. These instances include direct and inverse pose determination in general parallel (e.g., Stuart-Gough platform) and serial (e.g., 6R) robots, singularity distribution and workspace mapping. This has also been carried out in cases of lower degree of mobility parallel robots as well as for planar and spherical mechanisms. Fundamental to such formulations is the algebraic parametrization of the various displacement groups (planar, spherical, spatial). These parameters are usually elements of the group's quaternion algebra. We contend that this approach provides a most effective insight into the structure of the equations and what it reveals about the corresponding mechanical systems under investigation.

Topics to be addressed are:

- Methods to establish the sets of equations – the canonical equations,
- Solution methods for sets of polynomial equations,
- Jacobian and singularities,
- Some examples.

Key words: Kinematics, analysis and synthesis of mechanisms, singularities

MSC 2010: 53A17, 70B15.

On János Bolyai and His Appendix, "From nothing I have created a new different world"

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With the above sentence János Bolyai, the young military architect engineer indicated his discovery to his father, Farkas (Wolfgang) Bolyai in his famous letter on November 3^{th} 1823, Temesvár (Timişoara). The occasion was, very probably, that he found the very elegant formula

$$\exp\left(\frac{x}{k}\right) = \cot\left[\frac{\Pi(x)}{2}\right]$$

between the parallel distance x and its parallel angle $\Pi(x)$. Here k is the universal positive constant, caracterizing the whole system S (the hyperbolic geometry). In the system Σ (i.e. in the Euclidean geometry) if k tends to the infinity (∞), then $\Pi(x)$ tends to $\frac{\pi}{2}$.

In this talk I will sketch the 7 steps of his synthetic proof, which lead to this formula. As byresults, he models the classical Euclidean plane geometry on the surface **F** (horosphere) of his absolute space geometry, he derives the absolute sine theorem for a rectangle, he proves that the spherical geometry is absolute. I will only indicate some further parts of 'these most extraordinary two dozen pages of the history of thinking', as G.B. Halsted wrote in the preface of his first English translation of the Appendix.

Key words: János Bolyai, absolute geometry, parallel distance, parallel angle

MSC 2010: 51M05, 51M10

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E-learning in constructive geometry and graphics at Faculty of Civil Engineering in Rijeka

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Starting from competences that students need as future engineers, we define the outcomes of the learning process. (Based on http://www.euceet.utcb.ro/).

Mixed (hybrid) system of e-learning can help in that. The live (f2f) and e-component of teaching combines the advantages of both systems: the multimedia presentation materials, forums, online tests, chat, wiki-activities to all classic methods, which affect the quality of education. Strategy for the introduction of e-learning is influenced by analysis of problems, encountered in teaching.

This presentation will show experience with e-learning technology, as the e-component in geometric courses at Civil Engineering Faculty in Rijeka.

E-course has a thematic structure. Each chapter has a concise script, interactive graphic examples, preparation of exercises and tasks for training and a forum for discussion. There are examples solved step by step, photos of engineering objects, student works and a few humorous attachments. For each exercise there is a short online test. Program-tasks, after the drawing, must be modeled in CAD and electronically submitted (uploaded) for evaluation and feedback.

One of basic competences for modern civil engineering student is the engineering graphics literacy - "reading and writing" using engineering language. That includes CAD software and 3D modeling.

The bonus-activities are not compulsory but are well accepted by students. Self-evaluations are offered by chapters, as well as online test exam et the end. Regular tests and exams have the online part. The results are obtained quickly and can be seen through the e-course.

Monitoring of that whole process is based on constant moderation of the course and supported by questionnaires.

Ball packings in Thurston geometries

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Finding the densest (not necessarily periodic) packing of spheres in the 3-dimensional Euclidean space is known as the Kepler Problem: *No packing of spheres of the same radius has a density greater than the face-centered (hexagonal) cubic packing.* This conjecture was first published by Johannes Kepler in his monograph *The Six-Cornered Snowflake (1611)*, this treatise inspired by his correspondence with Thomas Harriot (see Cannonball Problem). In 1953, László Fejes Tóth reduced the Kepler conjecture to an enormous calculation that involved specific cases, and later suggested that computers might be helpful for solving the problem and in this way the above four hundred year mathematical problem has finally been solved by Thomas Hales. He had proved that the guess Kepler made back in 1611 was correct.

In mathematics, sphere packing problems concern the arrangements of non-overlapping identical spheres which fill a space. Usually the space involved is three-dimensional Euclidean space. However, ball (sphere) packing problems can be generalized to the other 3-dimensional Thurston geometries.

In an n-dimensional space of constant curvature $\mathbf{E}^n$, $\mathbf{H}^n$, $\mathbf{S}^n$ $(n \geq 2)$ let $d_n(r)$ be the density of $n+1$ spheres of radius r mutually touching one another with respect to the simplex spanned by the centres of the spheres. L. Fejes Tóth and H. S. M. Coxeter conjectured that in an n-dimensional space of constant curvature the density of packing spheres of radius r can not exceed $d_n(r)$. This conjecture has been proved by C. Roger in the Euclidean space. The 2-dimensional case has been solved by L. Fejes Tóth. In a 3-dimensional space of constant curvature the conjecture has been settled by Böröczky and Florian in [2], and it has been proved by K. Böröczky in [1] for n-dimensional spaces of constant curvature $(n \geq 4)$. In [3] and [9] we have studied some new aspects of the horoball packings in $\mathbf{H}^3$.

The goal of this talk is to generalize the above problem of finding the densest geodesic ball (or sphere) packing to the other 3-dimensional homogeneous geometries (Thurston geometries)

$$\widetilde{\mathbf{SL_2R}},\ \mathbf{Nil},\ \mathbf{S}^2\!\times\!\mathbf{R},\ \mathbf{H}^2\!\times\!\mathbf{R},\ \mathbf{Sol},$$

(see [5], [6], [7], [8]) and to describe a candidate for the densest geodesic ball arrangement. The greatest density until now is ≈ 0.85327613, whose horoball arrangement is realized in the hyperbolic space $\mathbf{H}^3$. In this talk we show a geodesic ball arrangement in the $\mathbf{S}^2\!\times\!\mathbf{R}$ geometry whose density is ≈ 0.87499429.

E. Molnár has shown in [4] that the homogeneous 3-spaces have a unified interpretation in the projective 3-sphere $\mathcal{PS}^3(\mathbf{V}^4, \boldsymbol{V}_4, \mathbb{R})$. In our work we shall use this projective model of each Thurston geometry.

Key words: Thurston geometry, ball packing, Dirichlet-Voronoi cell

MSC 2010: 52C17, 52C22, 53A35, 51M20

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Contributed talks

The parabola in universal hyperbolic geometry

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We will present some new results and formulas for the parabola in hyperbolic geometry, and illustrate them with GSP diagrams. We use the framework of Universal Hyperbolic Geometry developed by N J Wildberger (UNSW), which extends the projective model of Beltrami and Klein, and in which the outside of the disk plays an equal role to the inside. The metrical notions are quadrance and spread, hyperbolic variants of the corresponding notions of Euclidean rational trigonometry. We will see that many classical properties of the Euclidean parabola can be extended to the hyperbolic setting, and also illustrate interesting differences between the two.

Interlaced motifs by one single strip or chain

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Probably the most exciting examples of rhytmical (periodic) decorations and artworks are those which provide the expression of being woven. These interlaced patterns can be investigated also in a strict mathematical way, with symmetry properties in focus. Examining - for instance - Celtic knotworks, the need for complete classification of the woven strip ornaments arose [1].

As it was pointed out [2], the topic is strictly connected with the theory of color groups, more precisely with the black and white frieze, rosette and plane groups, respectively. In [3] P.R.Cromwell gave the complete enumeration of different types of the 2-sided rosettes, frieze and of periodic patterns, where the base of the classification of the interlaced patterns were henomerism: a bijection between the set of copies that acts compatible with all symmetries of the motifs.

In this presentation we examine the following question: which woven pattern-types are realisable by using one single strip or a chain?

Key words: interlaced pattern, symmetry, henomerism

MSC 2010: 52C99

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Comparative survey using Mental Cutting Test

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In the school year 2010/2011 a comparative survey has been done at our University using the well-known Mental Cutting Test. We made civil engineering and architecture students fill the test in the beginning and at the end of the autumn semester. The test results have been statistically evaluated. We found interesting connections between the effects of faculty, sex, handedness, previous studies and notes in DG, and the results in MCT.

Key words: Mental Cutting Test, descriptive geometry

MSC 2010: 51N05

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Some of the most beautiful theorems in triangle geometry

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In the course of this presentation the envelope of the Wallace - Simson lines of triangle will be presented. It will be also shown that in a real projective plane, all vertices in a pencil of parabolas lie on a bicircular quartic. That quartic will be constructed.

Key words: projective plane, triangle, pedal curves, Wallace's (Simson) line, quartic

MSC 2010: 51N15, 51M15

The general case of main axis for 2^{nd} order cone

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The main objective of this paper is determination of the three main axes of a second order cone (three mutually orthogonal axes) in a constructive geometric manner. We consider three characteristic cases determined by three different types of the base plane section, i.e., the ellipse, hyperbola or parabola. The constructive procedure is based on polarity and involutory spatial mapping. Each 2^{nd} order cone has its orthogonal auto-polar tetrahedron, in such position that one of its 4 sides is the infinite plane of 3D space.

The setting is as follows. The apex of the cone (general case) coincides with another apex of a right circular cone with the same height and common base plane. The radius of its base circle is equal to its height. These two cones define, in space, two collocal bundles of lines/planes. The intersections of two bundles of lines/planes are the main axes/planes of symmetry of cone.

The solution of the problem lies in application of a correlative mapping of lines (generatrices of the cone) and points (each point in base curve), i.e., lines and planes in 3D space. The spatial intersection of two bundles of lines/planes is provided by correlative, i.e., collinear, base curve planes of both cones.

Key words: 2^{nd} order cone, polarity, auto-polar tetrahedron, correlative planes, main axis of cone

MSC 2010: 51N05

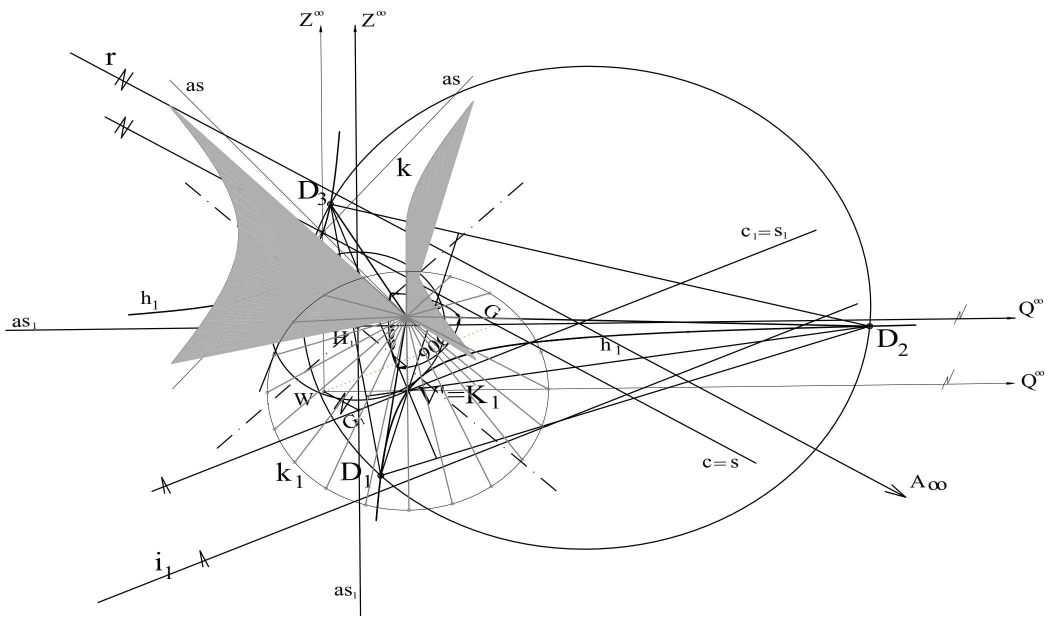

Figure 1: Three main axis of 2^{nd} order cone: case with base curve-hyperbola

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Pedagogical role of mathematical proof for students at non-mathematical faculties

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Mathematical proof plays a crucial role in mathematic-oriented education, i.e. education of mathematicians. As for non-mathematical areas (informatics, economy, programming, architecture, etc.) applying mathematical tools and procedures is important element of their education. But what about using proof and proving in those areas of education? They become important if one looks on proving as an activity in mathematics education which serves to elucidate ideas worth conveying to the student (Hana, [1]). In order to get some answers we have investigated several research questions: To what degree and in what ways are students able to construct proof in classroom? Do they recognize a proving method used in a theorem (direct proof, contra positive claim, counterexample, etc.)? Can students identify main ideas of a proof and explain their meanings? Do students recognize which part of the theorem is a conjecture and which part is a consequence? What are their conceptions and attitudes about mathematical proof? To pursue these questions, two tests and a questionnaire were given to second and third year undergraduate students at Faculty of organization and informatics (approximately 200 students). The data were analyzed to identify and clarify some of the pedagogical roles of proof in education of non-mathematicians. We come to conclusion that not all students recognize or appreciate the importance of a proof but there are those who gain from it. The reason for that is because mathematics is not their core study, just an exam to pass. Also, they do not realize that ideas presented in proofs are helping them to enhance their logic and abstract way of thinking which can be very useful in business. Finally we can enhance students acceptance of proof if we analyze and explain benefits of a proof to students and if we connect it to other elements of course (like algorithms, applications etc.).

Key words: proof, pedagogy, non-mathematical study programs

MSC 2010: 97D70

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Creative dimensions III

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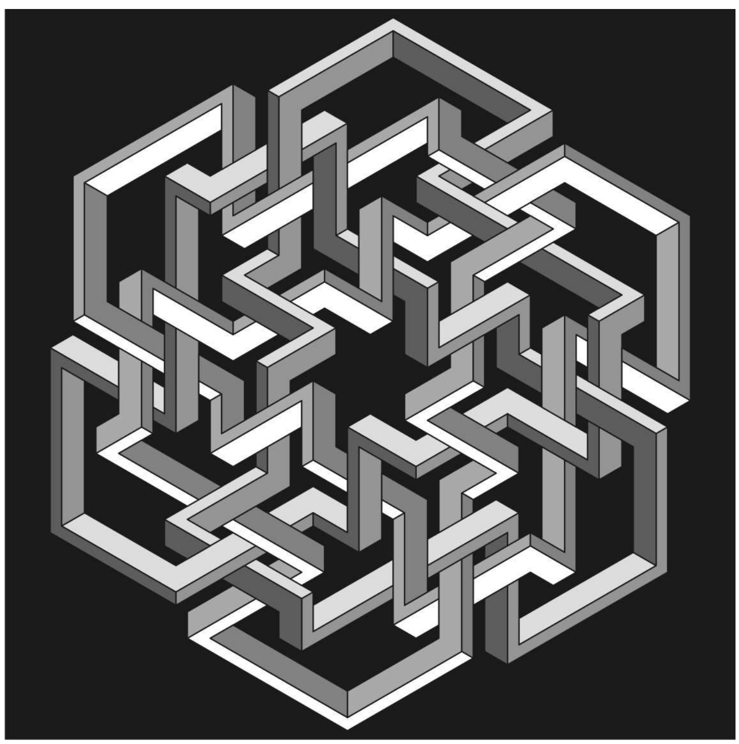
The goal of the talk is to present some new results of my research in modelling the sixth dimension.

On the plane you can examine a spatial object, made from 3D elements but you observe more side views simultaneously. The aim of the research is to develop the spatial imagination in students' visual education.

Beside the talk, an exhibition also makes the visual adventure complete.

Key words: higher-dimensional geometry in fine art

MSC 2010: 00A66, 97M80



Continental drift and oceanic conveyor belt - a geometrical pleading for the globe

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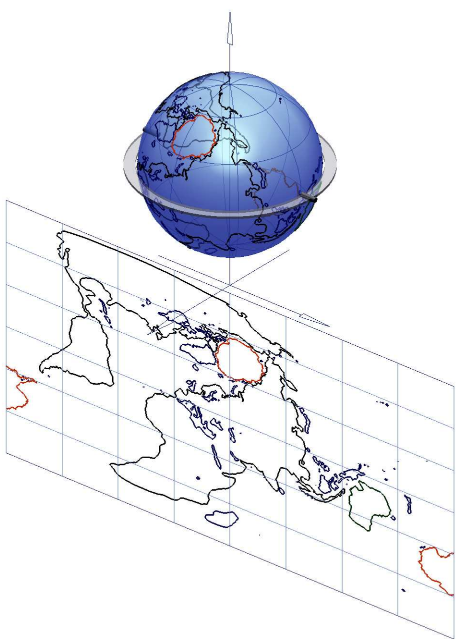
Even in the 21^{st} century, one can find illustrations of global issues on our planet in popular scientific and also scientific papers, where extremely distorting mappings of the sphere almost impede immediate understanding of the situation. One can see orbits of satellites or terminator lines that look like baggy sine curves or the "global oceanic conveyor belt" that surrounds the Antarctic several times such that one can see almost straight parts of the currents going from left to right several times. The Antarctic continent itself, parts of which perfectly fit to the Australian and the South American continent, appears as strangely shaped strip. This should not happen in the age of computational geometry.

We will show ostensive pictures and computer animations that demonstrate how understanding can be promoted by correct spherical geometry.



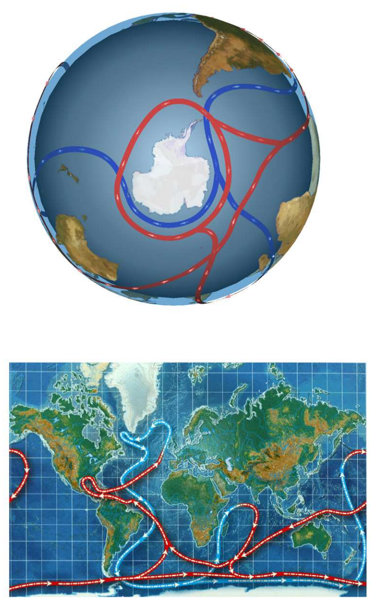


Figure left: Simple linear translation of spherical coordinates and the according rectangular map below.
Figure right: How the oceanic conveyor belt really works and below a typical publication.

Some triangle geometry of the triangle family

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In this work we consider a one-parameter triangle family $\mathcal{T}$. We prove that the set of the orthocenters, centroids, circumcenters, and some other sets of the triangle centers lie on different hyperbolae. Furthermore, it will be shown that the hyperbolae which are the sets of triangle centers that lie on the Euler lines of the triangle family $\mathcal{T}$ have a space interpretation with the Hohenberg transformation. In this presentation we also prove and construct two different cubics as the line envelopes of the side bisectors and Euler lines, and two different quartics as the circle envelopes of the circumcircles and Euler circles of the triangle family $\mathcal{T}$.

Key words: triangle centers, one-parameter family of triangles, envelope of lines, envelope of circles

MSC 2010: 51M04, 51M15

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On the thickness of $\langle p, q\rangle$ point systems

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Let p and q be positive natural numbers. The point set Σ^d in the d-dimensional space (d-space) of constant curvature is called a $\langle p, q\rangle$ point system if the following two conditions are satisfied.

1. There exists a real number $r \in \mathbf{R}^+$ such that an arbitrary open ball of radius r contains at most p points of Σ^d.
2. There exists a real number $R \in \mathbf{R}^+$ such that an arbitrary closed ball of radius R contains at least q points of Σ^d.

Let $r_p = \sup r$ and $R_q = \inf R$ be real numbers such that the point set Σ^d is a $\langle p, q\rangle$ point system. We say that Σ^d has the $\langle p, q\rangle$ property. The quotient $\dfrac{r_p}{R_q}$ is called the $\langle p, q\rangle$ thickness of Σ^d. Let

$$\kappa(\Sigma^d; p, q) = \frac{r_p}{R_q}. \tag{1}$$

For the given numbers d, p, q we consider

$$\kappa(d; p, q) = \sup \kappa(\Sigma^d; p, q), \tag{2}$$

where the supremum is taken over all $\langle p, q\rangle$ point systems.

The problem is the determination of $\langle p, q\rangle$ point systems with thickness $\kappa(d; p, q)$. In case $p = 1, q = 1$ we have the so called (r, R) point systems defined by Delone [1]. For characterisation of (r, R) point systems Ryskov [3] used the reciprocal of (1) and called it the density of the (r, R) point system. L. Fejes Tóth [2] defined the closeness of packings of balls and the looseness of coverings with balls. Ryskov and L. Fejes Tóth gave different formulation of the same problem. The $\langle p, q\rangle$ thickness of point systems, the closeness of multiple packings, the looseness of multiple coverings are the generalizations of the above problems and are different from each another.

In this lecture we review the most important results and give lower and upper bounds for $\kappa(d; p, q)$.

Key words: $\langle p, q\rangle$ point system, thickness

MSC 2010: 52C35

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Covering the unit square by some rectangles

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Let $f(d, n)$ be the smallest positive real number for which the d-dimensional unit cube can be covered by n rectangular boxes of diameter at most $f(d, n)$. We will give the exact values of $f(2, 5)$ and $f(2, 6)$.

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Geometry at the technical faculties in Croatia

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This is an introductory summary of the comparative research that is being carried out by the members of CSGG and Faculty of Science - Department of Mathematics. It is focused on the new teaching aspects of the geometry, [1], [9]. Beside resorting to computer that constitutes a great innovation for the teaching of geometry, over the past decade, many university teachers faced a serious educational problem since *having a good teaching of geometry at the university level necessary implies a good teaching at the primary and secondary schools*, [2], [3], [4], [5].

Our research is focused on the examination of the university content standards in geometry of five central European countries with similar problems as Croatia (Austria, Czech Republic, Germany, Hungary, Slovenia) in order to determine *the common problem-solution elements*, [6], [8], [7]. The results regarding the specific subject *Descriptive Geometry* (or similar) are presented. The analysis includes the following dimensions of the current subject curricula on the base of specific study program (only for B.Sc. degree): aims and outcomes of the subject, curricular organization of the subject (separate subject or not, cross-curricular areas), status of the subject in the curriculum (compulsory, optional, etc.), total number of lessons and conditions of implementation of the curriculum.

Key words: descriptive geometry, spatial ability, dynamic geometry, education

MSC 2010: 97B10, 97B40

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Steiner's ellipses of the triangle in an isotropic plane

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The concept of Steiner's ellipse and Steiner's point of a triangle in an isotropic plane will be introduced. A number of statements about relationships between the introduced concepts and some other geometric concepts about triangle will be investigated in an isotropic plane.

Key words: Steiner's ellipse, Steiner's point, standard triangle

MSC 2010: 51N25

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Geometry and architecture of triangles

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A triangle is a figure which has been attracting the mathematicians and artists of different fields for thousands of years. Due to its geometric characteristics, its symbolic meaningfulness and possibility of further divisions it was already asserted in early architectures, regardless of their spatial restraints. In the architecture today triangle is being used less like a symbol but more because of its excellent static characteristics. The contribution wants to show the meaning of triangle within architecture, its transformations, and how it has been used in the past and, of course, today.

Key words: architecture, geometry, triangle

On the geometry of B-scrolls

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The theory of ruled surfaces in Lorentz-Minkowski 3-space is to a large extent like in Euclidean geometry. The differences clearly come from the existence of null vectors. So of special interest are ruled surfaces the rulings of which have null direction.

A subclass of these surfaces are B-scrolls first introduced by L. K. Graves (1979) in the context of isometric immersions of the Minkowski 2-plane in the Minkkowski 3-space. There are many investigations on B-scrolls but rare discussions of there geometry. The presentation tries to improve the knowledge taking the euclidean and affine point of view in account.

Key words: Lorentz-Minkowski space, ruled surface, B-scroll, Cartan frame, null curve

MSC 2010: 53A05, 53B30

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On the use of digital tools in perspective images generation

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In architectural practice perspective images (PI) play an important role to simulate the real space. However, three point perspectives are sometime inconvenient, particularly when metrics and spatial relationships are to be detected from PI. Although computer modeling is useful in the understanding of architectural form, the final models which are generated using digital tools do not always give fully adequate solution by means of its mode of perspective. The problem might be detected in software's allowance of totally free choice of a view point. Thus, very sharp three point perspectives may be obtained. Since our students in their final work widely use computer aided perspective generation we often meet "bad" PI from which one can neither easily detect the spatial relationships nor calculate metrical properties. The solution to the problem we see in creating plug-ins for 3D modeling software which provide proper choice of a view point, that is, the loci of view points which divide acceptable from unacceptable perspective presentations. This restriction, either as a forbidden view point disposition or as a warning to the user, is useful when complex architectural composition are to be rendered, particularly in educational purposes. In this paper, through the restitution of an inadequate PI, we discuss the generation of loci of view points with acceptable PI using simplified complex contour.

Key words: perspective images, restitution, plug-ins for 3d modeling software, education

Kiepert conics in Cayley-Klein geometries revisited

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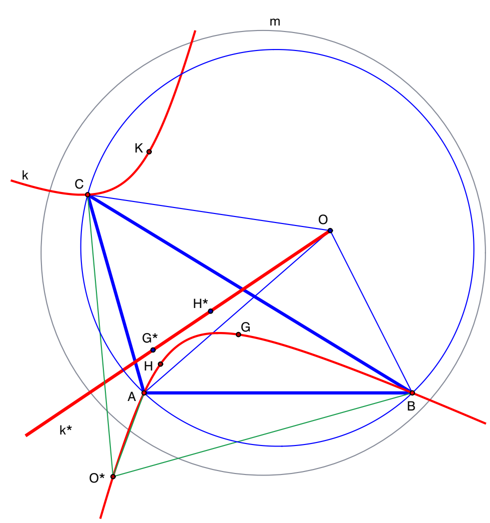
We consider the *Cayley-Klein* model of the hyperbolic plane where arguments from projective geometry can be used. Let ABC be an arbitrary triangle in the hyperbolic plane. A triangle $A'B'C'$ is called a hyperbolic *Kiepert triangle* with respect to ABC if $A'BC$, $AB'C$, and ABC' are hyperbolical isosceles triangles constructed externally (or internally) on the sides AB, BC, and CA. The triangle ABC and $A'B'C'$ are perspective from some centre K and the locus of K is the first *Kiepert conic* k in the hyperbolic plane. The conic k is also determined as the conic through the vertices A, B, C, the hyperbolic centroid G, and the hyperbolic orthocenter H of the triangle ABC. The image of the first Kiepert conic k under the hyperbolically isogonal transformation with fundamental points A, B, C is the hyperbolic *Bocard axis* $k^\star$ of ABC. In this study a proof of this fact will be given.

In the Euclidean geometry this result is quite obvious, because the orthocenter H and the circumcenter O of a triangle ABC are a pair of isogonal points. This is not true in the hyperbolic geometry and we have to show that the isogonal point $O^\star$ is a point on the first Kiepert conic in hyperbolic geometry.

Key words: Cayley-Klein geometries, triangle, isogonal transformation, Kiepert conics

MSC 2010: 51N05, 51N15, 51F99



Remarks on the local geometry of a discrete Bäcklund configuration

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We study two cyclic sequences $(a_i, \mathbf{m}_i)$ and $(b_i, \mathbf{n}_i)$, $i = 0, \dots, 3$ of contact elements (points plus unit normal vectors) such that any two neighbouring contact elements have a common tangent sphere. This configuration is of interest because it describes the local geometry of two pseudospherical Bäcklund mates in discrete curvature line parametrization. It is full of curious geometric relations: The four points a_0, a_1, a_2, a_3 and the four points with coordinate vectors $\mathbf{m}_0$, $\mathbf{m}_1$, $\mathbf{m}_2$, $\mathbf{m}_3$ form circular quadrilaterals Q_a and $Q_{\mathbf{m}}$, respectively, the four lines spanned by the four contact elements $(a_0, \mathbf{m}_0)$, $(a_1, \mathbf{m}_1)$, $(a_2, \mathbf{m}_2)$, $(a_3, \mathbf{m}_3)$ form a skew quadrilateral on a hyperboloid of revolution etc.

We explore some new geometric properties of this figure with the ultimate aim of simplifying a computer-assisted proof for the formula

$$\frac{A_0}{A} = -\frac{\sin^2\varphi}{d^2}$$

that relates the oriented areas A_0 and A of Q_a and $Q_{\mathbf{m}}$, respectively, with the distance d of a_i and b_i and the angle φ between $\mathbf{m}_i$ and $\mathbf{n}_i$.

Key words: Discrete curvature line parametrization, rotation, reflection, pseudosphere, Bäcklund transform

MSC 2010: 51F25, 53A17, 53A05

B-spline surface patches on triangle meshes

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In this paper we present a construction method of B-spline surface patches approximating a triangle mesh locally. Triangle meshes are frequently used discrete surface representations in computer aided modeling systems. The input data of this patch construction are curvature values estimated at specified triangles of the mesh and boundary data. The constructed B-spline patch is of 4x4 degree. Its control points are computed by a circle approximation method and by using boundary conditions. The error analysis is made on "synthetic" meshes generated by triangulating analytic surfaces.

The constructed patches replacing well defined regions of the mesh can be used in different applications, e.g. for mesh decimation algorithms or rendering.

Key words: B-splines, triangular meshes, surface approximation

MSC 2010: 65D07

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Covering the unit d-cube by n convex bodies of minimal k-energy

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The energy of a finite sequence of points in $\mathbb{R}^d$ is the sum of squares of distances determined by the pairs of those points, that is

$$E(S) = \sum_{i=1}^{k} \sum_{j=i+1}^{k} \|p_j - p_i\|^2,$$

if $S = \{p_1, p_2, \ldots, p_k\} \subseteq \mathbb{R}^d$, and $k \geq 2$.

Let $k \geq 2$ be an integer. The k-energy of a convex body K in $\mathbb{R}^d$ is the maximal energy $E(S)$ of a sequence S of k points which are contained in K:

$$E_k(K) = \max\{E(S) \mid S = \{p_1, p_2, \ldots, p_k\},\ p_i \in K\ \forall i = 1, 2, \ldots k\}.$$

We examine the following problem: For given integers $n, k, d \geq 2$ what is the smallest possible value for the maximum of the k-energies of such n convex bodies which form a covering of the unit d-cube $[0,1]^d$ in $\mathbb{R}^d$? That is, the quantity

$$E(n,k,d) = \min\Big(\max_{1 \leq i \leq n} \big(E_k(K_i)\big) \mid [0,1]^d \subseteq \bigcup_{i=1}^{n} K_i, \\ K_i \text{ is a convex body for } i = 1, 2, \ldots n \Big)$$

is in question.

We show that if k, d are fixed and $n \to \infty$, then we can get better upper bounds for $E(n,k,d)$ than those obtained by any covering of $[0,1]^d$ by rectangular boxes. Our improvements are based on tilings of the plane by regular hexagons and on sphere coverings of the unit d-cube.

We also demonstrate the problem by dynamic geometry software for $d = 2$ and various values of k. In those special cases, this method provides numerical upper bounds for the problem as well.

As a consequence, we can give new upper bounds to another problem, estimating that at most how many points can be placed in a unit d-cube so that the distance between any two points is at least 1 (this problem, raised by Moser (1966), is considered in recent papers by Bálint and Bálint Jr. (2008), Joós (2010) and Talata (2010)). Now, we can improve on the known upper bounds for this problem in several dimensions.

Key words: covering, unit cube, packing, extremum problem

MSC 2010: 52C15, 52C17, 52C20, 52A40

Multiple circle arrangements

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A set of congruent open discs forms a p-fold packing or a p-packing in the plane if each point of the plane is an interior point of at most p of the discs. The fundamental question is the determination of the highest possible density and the circle arrangements with the extremal density, that is, the densest p-fold circe packings.

A covering of $\mathbf{R}^2$ with congruent closed discs is a q-fold covering or q-covering if each point of $\mathbf{R}^2$ belongs to at least q discs of the covering. The question is the determination of the minimal density and the circle arrangements with the minimal density, that is, the thinnest q-coverings.

The notation of multiple packing and multiple covering was introduced by L. Fejes Tóth.

A p-packing and a q-covering is called lattice packing and lattice covering if the centers of the discs in the p-fold packing and q-fold covering form a lattice.

We deal with results for lattice p-packing and q-covering.

Key words: multiple packing, multiple covering, density

MSC 2010: 52C15

Application of Bošković geometric adjustment method on five meridian degrees

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Josip Ruđer Bošković (Dubrovnik, 18th May 1711 – Milan, 13th February 1787) from his early scientific days began to publish theses on the issues of Earth's shape and size which represented a major scientific problem of the 18^{th} century. During the 18th century scientists were having a great discussion around the question whether the Earth was flattened or bulging at the poles. In the late 17^{th} century, Newton proved that the Earth should be flattened at the poles because of its rotation. Domenico Cassini assumed the opposite, that Earth had the shape of an egg so, at the end of 17^{th} and in the beginning of 18^{th} century he conducted comprehensive geodetic observations to prove his assumption. There existed two basic methods for determining the Earth's figure: pendulum experiments and the determination of the meridian arc length. The idea of the second method was to determine the length of the meridian arc that corresponded to one degree of latitude. French Academy carried out the measurements during 1730s to test theoretical interpretations of the Earth's figure.

Bošković came to the idea of confirming his assumption on meridians inequality by measuring the length of the meridian arc. To accurately determine the figure of the Earth, in his first attempt to determine ellipticity Bošković compared five arc lengths of one meridian degree, which he considered to be sufficiently accurate. Those were the measurements, of the meridian degrees, carried out in South America (Quito), South Africa (the Cape of Good Hope), France (Paris), Finland (the province of Lapland), and his own, carried out in Rome, Italy.

Whereas astronomical and geodetic measurements are liable with errors caused by various sources, Bošković was aware that the causes of errors cannot not be fully eliminated during the construction of instruments and measurements. When comparing mentioned five degrees of meridian, Bošković could not determine such an ellipsoid consistent with all the measurements. He decided to determine corrections that would fix all degrees and get a better estimate of true values.

In 1755 Bošković and Christopher Maire published the first results of those measurements and analysis of measured data in the book *De Litteraria Expeditione per Pontificiam ditionem ad dimentiendas duas Meridiani gradus et corrigendam mappam geographicam* (A scientific journey through the Papal State with the purpose of measuring two degrees of meridian and correcting a geographical map) on more than 500 pages. According to Bošković, data should be fixed in such a way that:

1. The differences of the meridian degrees are proportional to the differences of the versed sines of double latitudes
2. The sum of the positive corrections is equal to the sum of the negative ones (by their absolute values) and
3. The absolute sum of all the corrections, positive as well as negative, is the least possible one.

In his works Bošković gave geometric description of solutions for the mentioned conditions. In the paper we describe in detail the example with five meridian degrees. Data have been taken from Bošković original book. Geometric solution, described by Bošković himself, is not easy to understand at first, as it is noted by other authors who have studied the Bošković method as well. Today, by software for interactive geometry, his method can be analytically defined and visualized in a way which provides better understanding. For this purpose, GeoGebra has been used, a free mathematics software which joins geometry, algebra, statistics and calculus in one easy-to-use package.

Key words: Josip Ruđer Bošković, geometric adjustment method, GeoGebra

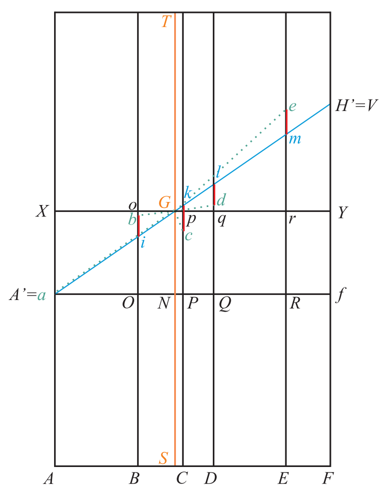

Figure 1: Bošković geometric adjustment method on five meridian degrees

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Projections of Projections with Common and Parallel Picture Planes

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A projection plane and a spatial shape in general position is given. (This shape is a cube for example in our presentation.) We construct two pictures by parallel or central projections on the given plane or ones parallel to it. We also need to know the shadow of the real or vanishing central point of the other projection on these pictures. These are the trace points of the common line of the projections' central points.

Two such pictures will be properly arranged on a common plane and we can gain new central or parallel projections of the given spatial shape on the same plane by two new planar projections. In special cases, we need to know the projection's central point of the pictures in space or the orthogonal shadows of those. This can be given easily in the case of a perspective picture of our cube. The shadow of the parallel projection lines of an axonometric projection can be defined on the base of the constructing method of Ferenc Kárteszi in general case [1].

The presentation proves the constructions by demonstrations of spatial interpretations of the different projections. A similar but more special constructing method of cubes' shadows was presented in a former lecture of the author [2].

Key words: constructive geometry, central and parallel projections, projections of projections

MSC 2010: 51N05, 51N15

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The backbone of the Earth

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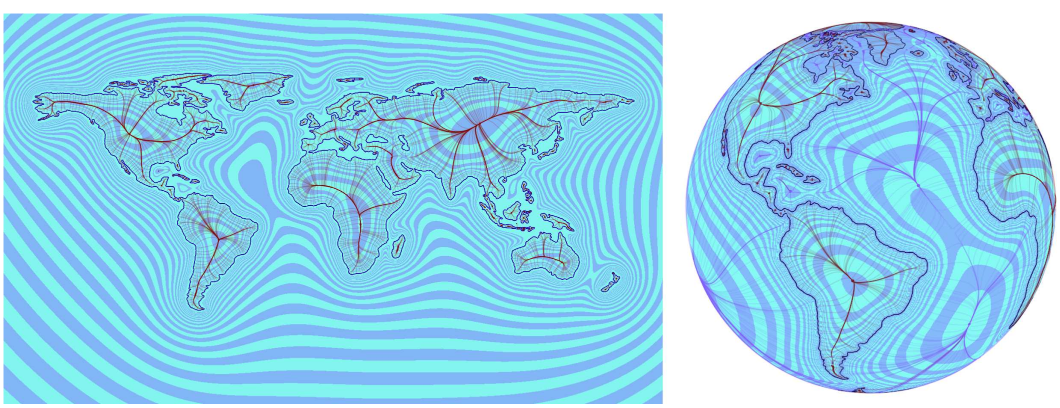
For a new book currently being written at our university the question of how a central path through a continent can be calculated was raised. Of course, this is a very vague question and many different solutions are possible. Going back to the seminal work of Blum [1] different methods have been proposed over the years, for instance, skeleton based approaches. Generally speaking, a skeleton of a shape A is a thin version of A that is equidistant to the boundaries of A. In literature several mathematical definitions of a skeleton (or topological skeleton) can be found, e.g., Gonzales and Woods [2] define the skeleton of A as the set of centers of the discs that touch the boundary of A in two or more locations.

In this lecture we will discuss our algorithm, which has been used to create the images of the Earth for the planar and spherical case as shown in the figure below. Especially in the depiction of the Earth it is necessary to perform the calculation on the sphere to obtain correct results. In either case the algorithm can be shortly summarized as follows: the algorithm starts by extracting the boundary of a 2D shape from an image and converting it into a vector outline. Along this outline negatively charged point charges are placed to create a repulsive force field (similar to the work of Cornea et al. [3]). Afterward, the trajectories of particles which are placed at uniform distances along the boundary are calculated and drawn with alpha blending. Finally, the equipotential lines of the force field are calculated.

Key words: topological skeleton, central path, vector field

MSC 2010: 00A66



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Posters

Entirely circular quartics in the pseudo-Euclidean plane

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A curve in the pseudo-Euclidean plane is circular if it passes through at least one of the absolute points. If it does not share any point with the absolute line except the absolute points, it is said to be entirely circular.

In this presentation, by using projectively linked pencils of conics, we construct all types of entirely circular quartics.

Key words: pseudo-Euclidean plane, entirely circular quartic, projectivity, pencils of conics

MSC 2010: 51M15, 51N25

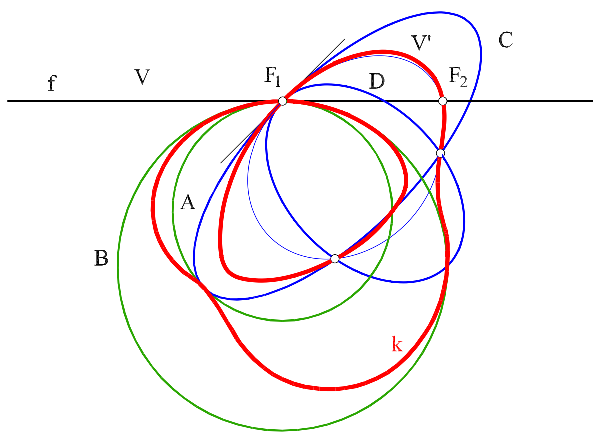

Figure 1: An entirely circular quartic of the type of circularity (3,1)

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