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ABSTRACTS

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Contents

Plenary lectures

1001 Images of Mathematics

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A series of partly new visualizations of mathematical theorems / proofs / ideas is introduced. A wide spectrum is covered, e.g., functions, formulas, curves and knots, planar geometry, topology, tessellations, fractals, non-Euclidean and higher dimensional geometry, kinematics, mapping theory and minimal surfaces. The talk provides a journey through the mathematical wonderland.

The far more than 1000 images are published in Georg Glaeser and Konrad Polthier: Bilder der Mathematik (Images of Mathematics), Spektrum Akad. Verlag / Springer, Heidelberg, May 2009.

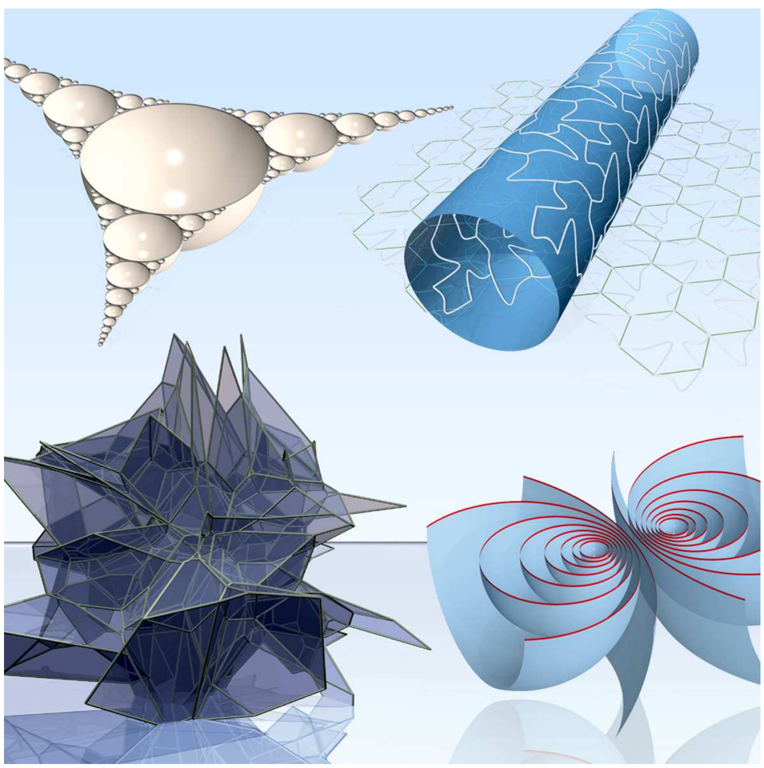

Figure 1

Algebraic Proofs of Napoleon-like Theorems

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We will consider idempotent medial quasigroups satisfying the identity $(ab \cdot b)(b \cdot ba) = b$. Surprisingly, geometric concepts such as equilateral triangles, centroids and midpoints can be introduced in this simple algebraic structure. An example are the points of the Euclidean plane with multiplication defined by $A \cdot B = C$, where C is the center of the equilateral triangle over $\overline{AB}$. In this setting the mentioned concepts have their usual geometric meaning.

The famous theorem attributed to Napoleon Bonaparte can be proved in this context: if equilateral triangles are erected over the sides of an arbitrary triangle, then their centers are the vertices of an equilateral triangle. We will prove some other similar theorems of plane geometry by using formal calculations in a quasigroup, and shed some light on what these theorems mean in settings different from the Euclidean plane. A representation theorem for this kind of quasigroups will also be explained.

Pairs of Tetrahedra with Orthogonal Edges

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The topic of our talk are pairs of tetrahedra in three-dimensional Euclidean space whose edges are orthogonal. We start with a number of examples from literature where this configuration already appeared. These include such diverse topics as the well-known "stella octangula", instantaneously flexible nets, the control net of Dupin cyclides and, naturally, the elementary theory of tetrahedra. Then we discuss several curious geometric configurations associated to orthogonal pairs of tetrahedra. Of particular interest is the case of intersecting edges. Simple examples are obtained by polarizing a tetrahedron circumscribing a sphere (a Koebe-tetrahedron). But non-Koebe pairs with orthogonally intersecting edges exist as well.

News on the Space-filling Zonotopes

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The lattice 3-model of any k-cube can be produced either as ray-groups based on symmetrical arranged starting edges or as sequences of bar-chains originated from a separate helix. We can replace the combinatorial method of counting of lower-dimensional j-elements of the k-cubes with tables and spatial constructions of unit elements. Some special models origin from regular and semi regular solids and the inner vertices determine solids which are similar to the initial ones.

Increasing the number of the sections in the bar-chains infinitely, continuous helices are created, whose sum can be called n-zonotope. We can connect the surface of our model and the hyperbolic surface rotated around the minor axis.

The suitable combinations of the models can result in 3-dimensional space-tiling. Our spatial tessellations can have fractal structure too. We can multiple the lengths of the solids' edges by addition of j-cubes in order to gain a similar solid to the original one. The further similar pairs can be constructed by these compound solids and so on. This means, that we can replace each space-filling mosaic with a similar one composed from the multiplied solids.

The general arrangement of the base edges was modified in case of odd k if we wanted to fill the space with the k- and j-models gained by the combination of the given edges. We found a new periodical tessellation based on the 3-model of the 9-cube without this modification. We can create interesting new space-filling mosaics in cases of $k = 5$ and $k = 7$ too originating the arrangement of the base edges from the symmetric 3-models of the 6- and 8-cubes.

The 2-dimensional shadows of the models and the sections of the above described mosaics yield unlimited possibilities to produce plane-tiling. The moved sectional plane(s) results in series of tilings or grid-patterns transforming into each other.

Key words: constructive geometry, 3-dimensional models of the hypercubes, tessellation

MSC 2000: 51M20, 68U07

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Contributed talks

Lacunary Polynomials and Blocking Sets

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A blocking set $\mathcal{B}$ is a set of points which meets every line of a projective plane. If each line of the plane contains at least t points of $\mathcal{B}$ then $\mathcal{B}$ is called a t-fold blocking set. This presentation describes their structure, existence and examines possible bounds on the size of blocking sets. Results of the theory of fully reducible lacunary polynomials (polynomials with a gap between its degree and second degree) are fundamental in determining further lower bounds on the size of blocking sets in Desarguesian projective planes.

Key words: blocking set, t-fold blocking set, lacunary polynomial

Geometrical Structure of the Geodesic Dome

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An analyzed structure of the geodesic dome is started by the geometry of two regular polyhedra (a regular icosahedron and a regular pentagonal dodecahedron). Faces and edges of the icosahedron and the dodecahedron are frequently subdivided.

These two type of polyhedra are circumsphered with the identical center of both of them. Faces and edges of these two type of polyhedra are first subdivided in wanted frequencies than they are projected from the polyhedra center on the sphere.

A projected struts length, derived from subdivided polyhedra, on sphere are newly designed.

The central angles between two vertices and same center of polyhedra and sphere are the most important, as well as mitre angles, knowing them it's simply to find out all the others sizes of the structure.

All the angles of the dome are independent of the size of the dome, only the length of struts depends on size of the dome.

The mitre angles are complementary with central angles.

Key words: icosahedron, dodecahedron, geodesic dome, sphere, central angle, projecting

Interrelation between Anaglyph Stereo Pairs

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In this paper anaglyph stereo pairs have been analyzed for the different position of viewpoints (the projection center) in one horizontal plane in compliance with the common projection plane.

In this manner gained anaglyph stereo pairs for competent position of viewpoints are in perspective collinear as well as in affine relation depending on the positions of the objects according to the projection plane.

Analyzed anaglyph stereo pairs detect the constructive procedure for direct creating of new anaglyph stereo pairs.

This paper also analyzed interposition of viewpoints and distance referring to the projection plane. The aim is to obtain clear and explicit anaglyphs. Constituted principles to consolidate direct modification and transformation of anaglyphs with inducted correlation of standard anaglyphs on contemporary media in manner to attain 3D illusion effect.

Key words: Anaglyphs, perspective, collineation, 3D illusions

Translation and helicoidal surfaces in the Galilean Space

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In this research we study translation and helicoidal surfaces in the special ambient space – the Galilean space ([4]). We are specially interested in the analogues of the results from the Euclidean space concerning Gaussian and mean curvature ([1], [2]).

A *translation surface* is a surface that can locally be written as the sum of two curves. A *helicoidal surface* (i. e. a generalized helicoid) is a surface obtained by the rotation of a profile curve around an axis and its simultaneous translation along the axis so that the speed of translation is proportional to the speed of rotation. When the profile curve is a straight line, a helicoidal ruled surface is obtained. Some results on helicoidal ruled surfaces can be found in ([3]). Rotation surfaces in the Euclidean space can be treated as helicoidal with no additional translation.

In the Galilean space we consider translation and helicoidal surfaces of constant Gaussian and mean curvature and find surfaces they are congruent with.

Key words: Galilean space, translation surface, helicoidal surface

MSC 2000: 53A35

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Student Attitudes towards Mathematics and Using New Technologies in Teaching Process

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In this presentation we will shortly describe a blended learning method within one mathematical course at the Faculty of Organization and Informatics, University of Zagreb where it is used virtual learning environment (VLE) Moodle. The course Mathematics 2 is taught as a blended (hybrid) course and it means that it is combining face to face teaching with VLE. Besides lectures and seminars in mathematics, students participate in peer group tutorials and use open source VLE Moodle.

The central point of the presentation will be analysis of the results of a questionnaire on students' attitudes towards mathematics and their evaluation of different aspects of technology enhanced learning. Finally, comments on correlation of students' attitudes towards mathematics and their success rate and fulfilling learning outcomes will be given.

It is essential to recognize the role that mathematical tools and models as well as students' attitudes towards mathematics play in a study program which is designed for a not-mathematical study program.

Key words: student attitudes, e-learning, hybrid learning

Designing Laminated Furniture in AutoCAD Software Environment

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Engineering Graphics as a subject at the Faculty of Forestry in Belgrade includes modeling of laminated furniture. Our students study methods and acquired skills of producing 3D objects. They are also taught how to use AutoCAD software tools, necessary for their presentation. This paper presents the process of projecting a chest of drawers, which is a task done by the students of Wood Processing. The importance of this paper lies in the presentation of the experience we have gained by introducing modeling, which helps the students to get a better perception of space and spatial relations.

The paper will also include the initial definition of drawers as blocks, the diversity of materials and their usage in the process of projecting a wardrobe (3D).

Key words: Engineering Graphics, 3D modeling, laminated furniture, software tools, AutoCAD

How to Ride a Hungry Donkey between Two Haystacks

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Let A and B be two given sets in the plane. The set of all points in the plane from which both sets A and B are seen at the same angle (and hence appear equally far) is called the **Buridan set** of the sets A and B. We determine the Buridan sets for some simple cases of A and B and discuss possible directions for future research.

Rose Surfaces

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We consider *roses* or *rhodonea* curves $R(m,n)$ which can be expressed by polar equations $r(\varphi) = \cos\frac{m}{n}\varphi$ or $r(\varphi) = \sin\frac{m}{n}\varphi$, where $\frac{m}{n}$ is a rational number in the simplest form. If $m \cdot n$ is odd, the curves close at a polar angle $\varphi = n \cdot \pi$ and have m petals. They are algebraic curves of the order $m+n$, with an m-ple point in the origin and with $\frac{1}{2}m(n-1)$ double points. If $m \cdot n$ is even, the curves close at a polar angle $\varphi = 2n \cdot \pi$ and have $2m$ petals. They are algebraic curves of the order $2(m+n)$, with a $2m$-ple point in the origin and with $2m(n-1)$ double points. [3, pp. 358-369]

For such curves we construct surfaces in the following way:

Let $P(0,0,p)$ be any point on the axis z and let $R(m,n)$ be a rose in the plane $z=0$. A rose-surface $\mathcal{R}(m,n,p)$ is the system of circles which lie in the planes ζ through the axis z and have diameters $\overline{PR_i}$, where $R_i \neq O$ are the intersection points of the rose $R(m,n)$ and the plane ζ.

We derive the parametric and implicit equations of $\mathcal{R}(m,n,p)$, visualized their shapes with the program *Mathematica* and investigate some of their properties such as the number and the kind of their singular lines and points.

Key words: roses, singularities of algebraic surfaces, *Mathematica*

MSC 2000: 51N20, 51N15, 51M15, 65D18

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Applications and Modifications of 3D - Voronoi Structures

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Complex space frames with respect to aesthetics and stability are the goal of a running project with architects on our university. Obviously there are many different ways to generate spatial structures, especially if randomness affects the generating process. One possibility is to use 3D-Voronoi structures as a starting point, which makes sense in terms of the frameworks load capacity. Inside an arbitrary bounding volume with predefined support points, a Voronoi tesselation is generated and then modified in several presented ways.

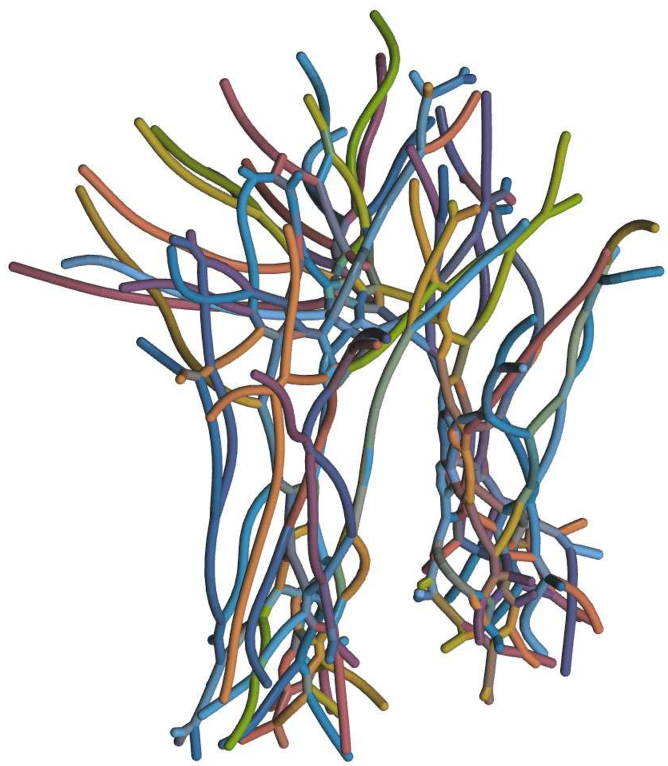

Figure 1: Structure derived from a Voronoi tesselation

Key words: voronoi tesselation, spatial structures, force directed algorithm

MSC 2000: 68U05

Isogonal and Isotomic Conjugates of a Triangle

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The positions and properties of a point in relation to its isogonal and isotomic conjugates are discussed with use of the dynamic program The Geometer's Sketchpad. Special attention is attached to the well-known triangle points as the four remarkable points of the triangle (incenter, circumcenter, orthocenter, centroid) and some other points as Gergonne point, Nagel point, Lemoine point etc. Finally formal properties of the isogonal and isotomic transformation are shown.

Key words: isogonal conjugate, isotomic conjugate, isogonal transformation, isotomic transformation

MSC 2000: 51M05

Approximation of Geodesics on Triangular Surfaces using Normals

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We present a novel method for approximation of geodesics on triangulations of smooth surfaces using the concept of Rotation minimizing vectors. In contrast to the concept of discrete geodesics on polyhedral surfaces the normal vectors at the vertices of a triangulation are additionally used. Piecewise linear approximation of points and normals lead to certain curves on the triangulation, which can be considered as approximations of geodesics. In each triangle these curves are analytically described by two ordinary nonlinear differential equations. We show that local the convergence of the points and interpolating (normal) lines of the triangulation to the points and normals of the reference surface implies the convergence of these curves to the associated geodesics on the surface.

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Isooptic Curve Associated to Couple of Second Order Curves

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In this paper the set of points (Isooptic curve) from which we can see two given curves of the second order under the constant angle are investigated. The equation and the graph of Isooptic curve associated to the couple of second order curves in different positions are given. Some properties of Isooptic curves for special case are also investigated.

Key words: Isooptic curve, parabola, ellipse, hyperbola

Blossoming or Polar Forms in deCasteljau and Oslo Algorithm

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Blossoming or polar forms brings nothing new in the spline theory, but it is the easy way to remember algorithms for polynomial splines. It is also useful for knot insertion algorithms. Here, the deCasteljau and Oslo algorithms will be shown in blossoming form.

Two Cobrocardial Heptagonal Triangles

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Some relations and statements concerning a heptagonal triangle will be studied in this lecture. The concept of the antiboutin triangle of the given heptagonal triangle will be introduced and some interesting relationships between these two triangles will be investigated. It will be also proved that the symmedian center, the Brocard diameter, the Brocard circle and the Lemoin line of the heptagonal triangle and its antiboutin triangle are coincident.

Inversion in Pseudo-Euclidean Geometry

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The notions of a circular and a totally circular curve in the pseudo-euclidean geometry, together with a notion of a curve circularity-type are introduced, and based on that, a complete classification of regular curves with respect to the general pseudo-eucliedean group of similarities $\mathcal{G}_4$ is given.

For the constructive geometry of the pseudo-euclidean plane, we choose a projective model given by the absolute figure $\{f, A_1, A_2\}$ consisting of a straight line f and two points $A_1, A_2 \in f$.

Also, an automorphic quadratic inversion is defined in the pseudo-euclidean plane and by using it all conditions for generating the circular cubics of certain degree and type have been found.

Key words: pseudo-euclidean plane, circular curve, automorphic inversion

MSC 2000: 51A05, 51M15

Remarks on the Minimal Surfaces of J.E.E.BOUR

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The minimal surfaces of BOUR are charcaterized by their local isometry to surfaces of rotation. The corresponding Weierstrass function is

$$F(z) = cz^{m-2} \ (c \in \mathbb{C}, m \in \mathbb{R})$$

where one can take $c = 1$ w.r.o.g. Several classical minimal surfaces belong to this class (catenoid and right helicoids ($m = 0$), or the Enneper surface ($m = 2$)).We point out several interesting geometric properties, most of them well known from classical papers.

Because of the isometry to surfaces of rotation the curves corresponding to meridians and parallels form an orthogonal net of zero and constant geodesic curvature respectively; they play an important role in the investigations.

We will also discuss properties of the evolute (focal) surfaces and certain involute surfaces the lines of curvatures of which are plane and spherical curves respectively.

Geometry in the National Curriculum Framework for Compulsory Education in Croatia

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In 2009 the development of the national curriculum framework for compulsory education in Croatia has been initiated. In the present proposal, mathematics is defined as a curriculum area. Themes from geometry are realized in the domains *Shape and space* and *Measurements*. In the talk, we will present the defined learning outcomes in these domains for the primary school, all secondary schools and for gymnasiums.

Key words: geometry, national curriculum framework

Projective Metric Visualization of the 8 Homogeneous 3-Geometries

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These so-called *Thurston geometries* are well-known. Here $\mathbf{E}^3$, $\mathbf{S}^3$ and $\mathbf{H}^3$ are the classical spaces of constant zero, positive and negative curvature, respectively; $\mathbf{S}^2\times\ \mathbf{R}$, $\mathbf{H}^2\times\mathbf{R}$ are direct product geometries with $\mathbf{S}^2$ spherical and $\mathbf{H}^2$ hyperbolic base plane, respectively, and a distinguished $\mathbf{R}$ -line with usual $\mathbf{R}$ -metric; $\sim\mathbf{SL_2R}$ and **Nil** with a twisted product of $\mathbf{R}$ with $\mathbf{H}^2$ and $\mathbf{E}^2$, respectively; furthermore **Sol** as a twisted product of the Minkowski plane $\mathbf{M}^2$ with $\mathbf{R}$. So that we have in each an infinitezimal (positive definite) Riemann metric, invariant under certain translations, guaranteing homogeneity in every point.

These translations are commuting only in $\mathbf{E}^3$, in general, but a discrete (discontinuous) translation group - as a lattice - can be defined with compact fundamental domain in Euclidean analogy, but some different properties. The additional symmetries can define crystallographic groups with compact fundamental domain, again in Euclidean analogy, moreover nice tilings, packings, material possibilities, etc.

We emphasize some surprising facts. In **Nil** and in $\sim\mathbf{SL_2R}$ there are orientation preserving isometries, only. In **Nil** we have a lattice-like ball packing (with kissing number 14) denser than the Euclidean densest one [5], [8]. In **Sol** geometry there are 17 Bravais types of lattices, but depending on an infinite natural parameter $N>2$ [6]. Except $\mathbf{E}^3$, $\mathbf{S}^3$, $\mathbf{S}^2\times\mathbf{R}$, $\mathbf{H}^2\times\ \mathbf{R}$ there is no exact classification result for possible crystallographic groups (!?).

Our projective spherical model [2] is based on linear algebra over the real vector space $\mathbf{V}^4$ (for points) and its dual $\boldsymbol{V}_4$ (for planes), upto positive real factor, so that the proper dimension is 3, indeed. A plane → point polarity or $\boldsymbol{V}_4\times\boldsymbol{V}_4\to\mathbf{R}$ scalar product (by specified signature) induces the invariant metric in a unified (nontrivial) way. We illustrate the topic some new pictures and animations mainly in $\sim\mathbf{SL_2R}$, **Sol** and **Nil** on the base of new publications, partly in preparation [1], [4], [6], [7], [9], [10].

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Generalized Gergonne Points

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The Gergonne point G of a triangle Δ in the Euclidean plane can be seen from the more general point of view, i.e., from the viewpoint of projective geometry. So it turns out that there is not a single Gergonne point associated with Δ: In general there are four of them. Since Nagel's point is the isotomic conjugate of G with respect to Δ we find four Nagel points associated with Δ. We reformulate the problems in a more general setting and illustrate the different appearances of Gergonne points in different affine geometries. There also appears a projective version of Darboux's cubic and finally a projective version of Feuerbach's circle appears.

Key words: triangle, incenter, excenters, Gergonne point, Nagel point, Brianchon's theorem, Darboux's cubic, Feuerbach's nine point circle.

MSC 2000: 51M04, 51M05, 51B20

On Generalized LN-Surfaces in 4-Space

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The present paper investigates a class of two-dimensional rational surfaces Φ in $\mathbb{R}^4$ whose tangent planes satisfy the following property: For any three-space E in $\mathbb{R}^4$ there exists a unique tangent plane T of Φ which is parallel to E. The most interesting families of surfaces are constructed explicitly and geometric properties of these surfaces are derived. Quadratically parameterized surfaces in $\mathbb{R}^4$ occur as special cases. This construction generalizes the concept of LN-surfaces in $\mathbb{R}^3$ to two-dimensional surfaces in $\mathbb{R}^4$.

Learner Oriented e-learning Against the Eternal Problem of Good Understanding

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A constructive geometry provides a precise theory of visualization but despite its clarity, students of all areas and all times at technical faculties have similar problems: a visualization, a knowledge acquisition and creative solving of space problems. As much as the teachers try, this problem is always present in some measure. So we started the research of student, more accurately, of always-present large group of students with this eternal problem. Many of them are encountering this geometry for the first time. Searching for the solution of this eternal problem, we are introducing e-learning in teaching, in mixed form at Civil Engineering Faculty (GF Rijeka). An example of new technologies for e-learning is the LMS (Moodle), which is applied in the last two academic years in the constructive geometry.

We explore the student's dilemmas and problems, the advices and an assistance they ask, frequently asked questions etc. We have elaborated hierarchy of causes and consequences of a major problem in order to offer within e-course the appropriate materials and activities to help at the right time.

We explore the needs of the profession. So the new applied tasks will enter in the e-course, from various areas of civil engineering. We expect the growth of understanding, motivation and quality of knowledge achieved.

One comes to interesting experiences and observations, which serve as a guideline and light on the way into the future. New technologies have enriched geometric classes at GF Rijeka for years but also brought new tasks and problems.

Curve of Foci of Pencil of Conics in Pseudo-Euclidean Plane

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A focus of an algebraic curve in the Euclidean plane is defined as an intersection of the isotropic tangents of the curve. The curve of class m has m^2 foci. An analogous definition is valid in the Pseudo-Euclidean plane. The pencils of point conics and line conics are studied on a model of PE-plane. Their curves of foci are constructed. It is shown that the curve of foci of the pencil of point conics is of order six. It has different degrees of circularity depending on the type of the pencil. It can be entirely circular as well. Some special cases that are not possible in the Euclidean plane are pointed out. The curve of foci of the pencil of line conics is circular cubic. It can happen to be entirely circular, which is not possible in the Euclidean case. Besides the curves of foci, for the same pencils of conics the curves of centers are constructed as well.

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Comments on Kokotsakis Meshes

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A Kokotsakis mesh is a polyhedral structure consisting of an n-sided central polygon p_0 surrounded by a belt of polygons in the following way: Each side a_i of p_0 is shared by an adjacent polygon p_i, and the relative motion between cyclically consecutive neighbor polygons is a spherical coupler motion. Hence, each vertex of p_0 is the meeting point of four faces.

These structures with rigid faces and variable dihedral angles were first studied in the thirties of the last century. However, in the last years there was a renaissance: The question under which conditions such meshes are flexible (infinitesimally or continuously) gained high actuality in the field of discrete differential geometry. The goal of this presentation is to extend the list of known continuously flexible examples (Bricard, Graf, Sauer, Kokotsakis) for n=4 by a new family, which includes the so far "isolated" case of flexible quadrangle-tesselations.

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Manipulating B-spline surfaces and their demonstration with Mathematica

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A B-spline surface is determined by its control points, the corresponding weights and the knot vectors in the two parameter directions. If all the weights equal to one, then the B-spline function representing the surface is polynomial. The shape of the surface can be manipulated in a direct way by changing these data. In applications a basic requirement is to generate B-spline surfaces satisfying prescribed boundary conditions. For the solution of the problem, how can the shape of the surface be manipulated according to boundary conditions, the control points along the border of the control net have to be computed from the given boundary data. First order boundary conditions, i.e. prescribed points and tangent vectors influence two rows, second order boundary conditions influence three rows of the control points in the control net.

We present a method developed for tube-shaped B-spline surfaces [see the reference], quadratic in the cross direction and cubic in the longitudinal direction, where the boundary conditions consist of one given closing point and the tangents of the longitudinal parameter curves at this point. According to this, the control net is extended by two rows of control points computed from the given boundary data. These are so-called phantom points, invisible for the user for further manipulations.

The Wolfram demonstration web-site allows to run the uploaded Mathematica programs interactively, and to change the values of the included parameters within prescribed intervals. The examples show the method of phantom points on interpolating B-spline curves with boundary conditions and on tubular B-spline surfaces with user-defined closing points.

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A Generalization of the Parallelogram Law for Affine Regular Polytopes

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Let P be a d-dimensional affine regular polytope, and let S be a $(d-1)$-dimensional section of P obtained as the intersection of P and an affine hyperplane. Denote by $F(P)$ those affine linear transformations f of $\mathbb{R}^d$ for which $f(P) = P$. Let $\{S_1, S_2, \dots, S_n\} = \{f(S) \mid f \in F(P)\}$. Then we show that

$$\sum_{i=1}^{n} Vol_{d-1}^2(S_i) = c(S,P) \sum_{j=1}^{m} Vol_{d-1}^2(F_j),$$

where $F_1, F_2, \dots, F_m$ are the facets of P, $Vol_{d-1}(\ )$ is the $(d-1)$-dimensional volume, and $c(S,P)$ is an affine invariant coefficient that depends only on P and S, fulfilling $c(S,P) = c(f(S), f(P))$ for any nonsingular affine linear transformation f. This statement is a generalization of a result of Yetter (2009), who obtained a similar equation for medial sections of a simplex, and it can be regarded as a generalization of the parallelogram law (stating $e^2 + f^2 = 2(a^2 + b^2)$ for a parallelogram having sides a, b and diagonals e, f) of elementary geometry as well.

Problems with Hemispherical Projections for Visibility Determination

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The generation of computer images from a three-dimensional representation requires a projection step to generate a two-dimensional image from the three-dimensional description. To provide an undistorted view this is usually done with linear transformations. Non-linear projections, though, are not common since they are hard to implement on current graphics hardware. However, they are useful for various algorithms, e.g., shadow maps for omnidirectional light sources, environment mapping or visibility textures.

In this presentation, we point out problems which we faced during the implementation of non-linear projections for our radiosity solver. The solver uses a hemispherical projection to determine the visibility from a specific point in the scene. This is beneficial since the complete half-space can be mapped to an image in one rendering step, instead of five rendering steps for a hemicube rendering. Non-linear projections in hardware, however, face the problem that only vertices are transformed by the graphics pipeline, which inevitably introduces errors in the projection. In the second part of the lecture possibilities to minimize these errors are discussed and compared to each other.

Key words: non-linear projections, visibility, global illumination.

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