Tomislav Došlić, Snježana Majstorović Ergotić, Luka Podrug (Eds.)

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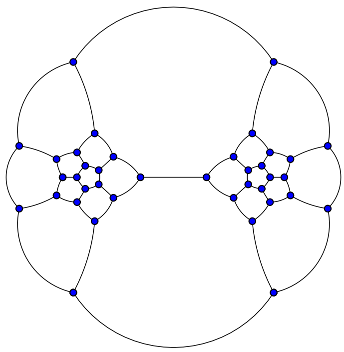

5th Croatian Combinatorial Days

Zagreb, September 19–20, 2024

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Preface

It has become something of a tradition – if repeating it three times makes it one – to begin by saying that writing the preface is always a pleasure for me. The last three prefaces started with that very sentence, and I see no reason to break the pattern now. It truly is a joy to witness the collective efforts of so many authors, reviewers, and editors come to fruition in this fifth volume of the CroCoDays conference proceedings.

Even more so, it is a pleasure to extend my sincere thanks to everyone who made this volume possible. First and foremost, I would like to thank the authors for sharing their research and choosing our Proceedings as their publication venue. In today's increasingly competitive academic environment, this choice is a meaningful vote of confidence, and their contributions are vital to the continued relevance and vibrancy of our conference series. I hope their work inspires readers to explore the results further and, hopefully, even initiate new collaborations.

I am also deeply grateful to our referees, whose timely and constructive reviews enhanced the quality of the accepted papers. Thanks are due as well to all conference participants – whether or not they submitted written contributions – for their presentations, discussions, and engagement, which helped make CroCoDays a lively and intellectually stimulating event.

Special thanks go to our Editor, Luka Podrug, for handling not only the technical preparation of this volume, but also many other behind-the-scenes details.

As in previous years, neither the conference nor this volume would have been possible without the generous support of our sponsors. I gratefully acknowledge the financial, logistical, and technical support of the Faculty of Civil Engineering. The publication of the Proceedings was partially funded by a grant from the Foundation of the Croatian Academy of Sciences and Arts. We also appreciate the Zagreb Tourist Board for providing promotional materials for our participants.

Finally, I thank ChatGPT for polishing my initial draft.

This volume is freely available on the conference website. Please feel welcome to share it with colleagues who might be interested in joining us in the future.

I look forward to seeing all of you—and many new faces—at the next CroCo-Days conference, scheduled for September 2026, and at many more conferences to come.

Zagreb, June 16, 2025 Tomislav Došlić

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A Note on the r-Stirling Numbers of the First Kind

Petra Marija De Micheli Vitturi and Ivica Martinjak

Abstract

We study a generalization of Stirling numbers characterized by a property of having elements 1 through r in different cycles of a permutation. Motivated by the known explicit formulae for the Stirling numbers of the first kind, we derive an explicit expression for the r-Stirling numbers. In addition, we establish an alternating sum identity involving these sequences of numbers.

Keywords: Stirling numbers, Pascal type array, permutation, binomial coefficients

1 Introduction

The Stirling number of the first kind, denoted by $\left[{n \atop k}\right]$, is the number of permutations of the set $\{1, 2, \dots, n\}$ with k disjoint cycles [7]. By placing a condition on certain elements, we obtain a generalized version of these numbers. For a positive integer r, the r-Stirling number of the first kind, denoted by $\left[{n \atop k}\right]_r$, is the number of permutations of the set $\{1, 2, \dots, n\}$ with k disjoint cycles such that the elements $1, 2, \dots, r$ appear in distinct cycles. Note that for $r = 1$, this definition recovers the Stirling numbers of the first kind, that is,

$$\begin{bmatrix} n \\ k \end{bmatrix}_1 = \begin{bmatrix} n \\ k \end{bmatrix}.$$

An explicit expression for $k = r$ follows directly from the definition, i. e. the number of permutations in the set $\{1, 2, \dots, n\}$ with r disjoint cycles

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such that the elements $1, 2, \ldots, r$ are in distinct cycles is equal to

$$\begin{bmatrix} n \\ r \end{bmatrix}_r = \frac{(n-1)!}{(r-1)!}. \tag{1.1}$$

Namely, we begin by placing the elements $1, 2, \ldots, r$ into r distinct cycles

$$\underbrace{(1 \ \cdots\ )(2 \ \cdots\ ) \quad \cdots \quad (r \ \cdots\ )}_{r}.$$

Next, we insert the remaining elements one by one. For element $r+1$, there are r possible positions. After $r+1$ is placed, to place element $r+2$ there are $r+1$ possibilities. Continuing in the same fashion, each new element has one more position available than the previous one, up to the element n, which has $n-1$ possible positions. By the product rule, the total number of such permutations is

$$r \cdot (r+1) \cdots (n-1) = \frac{(n-1)!}{(r-1)!}.$$

Furthermore, r-Stirling numbers of the first kind satisfy a recurrence relation

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_r = \begin{bmatrix} n \\ k-1 \end{bmatrix}_r + n \begin{bmatrix} n \\ k \end{bmatrix}_r \tag{1.2}$$

with

$$\begin{bmatrix} n \\ k \end{bmatrix}_r = 0 \text{ for } n < r$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_r = \delta_{k,r} \text{ for } n = r,$$

where $\delta_{k,r} =$ if $k \neq r$ and $\delta_{k,r} = 1$ if $k = r$. This recurrence follows by the same counting argument as the recurrence for the Stirling numbers of the first kind.

Using the recurrence relation (1.2), one can easily compute the values of $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_r$ for various values of n, k, and r. Tables 1 and 2 present the first few numbers for $r = 2$ and $r = 3$, respectively.

Santmyer introduced curious sequences of rational numbers and used them to derive relations between Stirling numbers of the first kind and generalized harmonic numbers [11]. Benjamin, Preston, and Quinn studied the set $\mathcal{T}_n$ of permutations of the numbers 1 through n into two disjoint, nonempty cycles. By enumerating this set, they obtained identities for Stirling numbers of the

$n\backslash k$	2	3	4	5	6	7
2	1	0	0	0	0	0
3	2	1	0	0	0	0
4	6	5	1	0	0	0
5	24	26	9	1	0	0
6	120	154	71	14	1	0
7	720	1044	580	155	20	1

Table 1: The 2-Stirling numbers of the first kind.

$n\backslash k$	3	4	5	6	7	8
3	1	0	0	0	0	0
4	3	1	0	0	0	0
5	12	7	1	0	0	0
6	60	47	12	1	0	0
7	360	342	119	18	1	0
8	2520	2754	1175	245	25	1

Table 2: The 3-Stirling numbers of the first kind.

first kind having the lower index $k = 2$. In particular, they proved that for $n \geq 1$,

$$\sum_{k=1}^{n} k \begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix},$$

providing a bijection between $\mathcal{T}_n$ and sets of the permutations into various numbers of cycles [2].

Pan studied Pascal-type arrays that generalize Stirling numbers of the first kind and Jacobi-Stirling numbers. The author presented a convolution identity for these sequences [10]. It is well known that Stirling numbers with a small value of the lower index k can be expressed by means of harmonic and hyperharmonic numbers. In particular,

$$\begin{bmatrix} n \\ 2 \end{bmatrix} = (-1)^n (n-1)! H_{n-1}$$

where $H_n := 1 + 1/2 + \cdots + 1/n$. Kowalenko followed this topic and presented the first 10 expressions of the Stirling numbers of the first kind in terms of the generalized harmonic numbers [9]. Recall that hyperharmonic numbers appear in the book by Conway and Guy [6].

There are several known generalizations of Stirling numbers. Carlitz introduced the weighted Stirling numbers [5]. The r-Stirling numbers are a special case of generalized Stirling numbers studied by Hsu and Shine [8]. Broder gave the generating functions for the r-Stirling numbers [4],

$$\begin{aligned}
\sum_{k=r}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_r x^k &= x^r \prod_{j=r}^{n-1} (x+j), \\
\sum_{n=k}^{\infty} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r \frac{x^n}{n!} &= \frac{1}{k!} \Big(\frac{1}{1-x}\Big)^r \Big[\ln \Big(\frac{1}{1-x}\Big)\Big]^k, \\
\sum_{k=0}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r \frac{x^n}{n!} t^k &= \Big(\frac{1}{1-x}\Big)^{r+1}.
\end{aligned}$$

Benjamin, D. Geabler, and R. Geabler have shown that harmonic numbers can be expressed in terms of r-Stirling numbers, which leads to many identities [1].

In what follows, we present an explicit formula for r-Stirling numbers in Proposition 2.1. We prove it employing combinatorial arguments. Our main result is presented in Section 3. Using mathematical induction, we give alternating sum identities in full generality.

2 Explicit formulae for small parameters

It is well known that there is a double-sum explicit formula for the Stirling numbers of the first kind,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{j=n}^{2n-k} \binom{j-1}{k-1} \binom{2n-k}{j} \sum_{l=0}^{j-n} \frac{(-1)^{l+n-k} l^{j-k}}{l!(j-n-l)!}$$

(there is no known single-sum formula, as there is for Stirling numbers of the second kind). The best-known simpler explicit formulas are those having the value of the lower index either small or close to the value of the upper index.

The following identities can be derived directly by enumerating permutations.

$$\begin{aligned}
\left[{n \atop n-1}\right] &= \binom{n}{2}, \\
\left[{n \atop n-2}\right] &= \frac{3n-1}{4}\binom{n}{3}, \\
\left[{n \atop n-3}\right] &= \binom{n}{2}\binom{n}{4}, \\
\left[{n \atop n-4}\right] &= \frac{15n^3-30n^2+5n+2}{48}\binom{n}{5}.
\end{aligned}$$

One can also apply the Newton-Girard formula for symmetric polynomials to get expressions for Stirling numbers with small values of the lower index. For $n \in \mathbb{N}$ we have

$$\left[{n \atop 2}\right] = (n-1)!H_{n-1}. \tag{2.1}$$

Furthermore, it holds true

$$\begin{aligned}
\left[{n \atop 3}\right] &= \frac{(n-1)!}{2}\Big((H_{n-1})^2 - H_{n-1}^{(2)}\Big), \\
\left[{n \atop 4}\right] &= \frac{(n-1)!}{6}\Big((H_{n-1})^3 - 3H_{n-1}H_{n-1}^{(2)} + 2H_{n-1}^{(3)}\Big),
\end{aligned}$$

where $H_n^{(m)} := \sum_{k=1}^{n} 1/k^m$. The recurrence for these expressions,

$$(m+1)\left[{n \atop m+2}\right] = \sum_{k=0}^{m} \left[{n \atop m-k+1}\right] H_{n-1}^{(k+1)},$$

is given by Shen [12].

There is a nice proof for the relation (2.1) by a counting argument. We consider the set of permutations of length $n+1$ with two cycles and address the question of what number of permutations that have r as the smallest element in the right cycle is, where $2 \le r \le n+1$. Thus, we have

$$(1 \ \cdots \ \cdots \ )(r \ \cdots \ \cdots \ )$$

as the structure of the permutations. Having known that r is the smallest element in the right cycle, we conclude that the left cycle contains numbers $1, 2, \ldots, r-1$. Therefore, there are

$$(r-2)!$$

possibilities to arrange these numbers. Now, the element $r+1$ can be either in the left or the right cycle. There are r possibilities to place it, since $r+1$ can be placed to the right side of any of the elements 1 through r. Furthermore, we have $r+1$ possibilities to arrange the element $r+2$. In the same manner, one concludes that there are

$$r(r+1)(r+2)\cdots n = \frac{n!}{(r-1)!}$$

ways to arrange numbers $r+1, \dots, n$.
Finally, by the product rule, there are

$$(r-2)!\frac{n!}{(r-1)!} = \frac{n!}{r-1}$$

considered permutations. Summation on r completes the proof,

$$\begin{aligned} \begin{bmatrix} n+1 \\ 2 \end{bmatrix} &= \sum_{r=2}^{n+1} \frac{n!}{r-1} \\ &= n! \sum_{r=2}^{n+1} \frac{1}{r-1} \\ &= n! \sum_{k=1}^{n} \frac{1}{k} \\ &= n! H_n. \end{aligned}$$

Proposition 2.1. *For positive integers n, we have the following explicit formula for the r-Stirling numbers:*

$$\begin{bmatrix} n \\ n-1 \end{bmatrix}_r = \frac{(n-r)(n-1+r)}{2}.$$

Proof. We consider the set of permutations of the set $\{1, \dots, r, \dots, n\}$ with $n-1$ cycles such that the elements $1, \dots, r$ are in distinct cycles, that is

$$\underbrace{(1\ \cdots\ )\quad \cdots \quad (r\ \cdots\ )(\ \cdots\ )\quad \cdots \quad (\ \cdots\ )}_{n-1}.$$

Out of $n-1$ cycles, r are reserved for the elements $1, 2, \dots, r$, each in its own cycle. Thus, there are $n-1-r$ remaining cycles. To form these remaining cycles, we choose $n-1-r$ elements from the remaining $n-r$ elements, which can be done in $n-r$ ways, leaving one element unplaced. We denote this remaining element by a.

Now, a can be inserted into one of the existing cycles, either into one of he r cycles with element $1, \dots, r-1$ or r, or into one of the remaining ones. In the first case, there are r possibilities.
In the other case, a can be placed into the remaining $n-1-r$ cycles. Let x denote the element currently in the cycle into which a is placed. Since inserting a into the cycle of x is equivalent to inserting x into the cycle of a, such configuration is counted twice. Therefore, this contributes $\frac{n-1-r}{2}$ valid placements.
Therefore, the total number of such permutations is equal to

$$(n-r)\left(r+\frac{n-1-r}{2}\right)=\frac{(n-r)(n-1+r)}{2}.$$

□

3 The alternating sum

The number of permutations of n elements with an even number of cycles is equal to the number of those with an odd number of cycles. Namely, in the standard notation, the element 2 is either in the first cycle or it is the first element in the second cycle. This provides the involution

$$\begin{gathered}(1\, a_1 \;\cdots\; a_j\, 2\, b_1 \;\cdots\; b_k)(\;\cdots\;)\quad\cdots\quad(\;\cdots\;)\\ \uparrow\downarrow\\ (1\, a_1 \;\cdots\; a_j)(2\, b_1 \;\cdots\; b_k)(\;\cdots\;)\quad\cdots\quad(\;\cdots\;)\end{gathered}$$

which changes the parity of a permutation [3]. This reasoning proves that the alternating sum of Stirling numbers with the upper index n is equal to zero,

$$\sum_{k=1}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}=0.$$

We now present an alternating sum identity involving r-Stirling numbers of the first kind and provide a proof using strong induction.

Theorem 3.1. *For positive integers n and $r\geq 2$, r-Stirling numbers of the first kind satisfy*

$$\sum_{k=r}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_r=(-1)^r\frac{(n-2)!}{(r-2)!}. \tag{3.1}$$

Proof. The relation (3.1) holds true for $n = r$, since the right hand side of the equation gives $(-1)^r$ which is equal to

$$(-1)^r \begin{bmatrix} r \\ r \end{bmatrix}_r.$$

Moreover, for $n = r + 1$, the left-hand side of the equation gives

$$(-1)^r \begin{bmatrix} r+1 \\ r \end{bmatrix}_r + (-1)^{r+1} \begin{bmatrix} r+1 \\ r+1 \end{bmatrix}_r = (-1)^r (r-1)$$

by relation (1.1), while the right-hand side of the equation gives

$$(-1)^r \frac{(r+1-2)!}{(r-2)!} = (-1)^r (r-1).$$

Furthermore, by applying the recurrence relation (1.2), we have

$$\begin{aligned}
&\sum_{k=r}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_r \\
&= (-1)^r \begin{bmatrix} n \\ r \end{bmatrix}_r + (-1)^{r+1} \begin{bmatrix} n \\ r+1 \end{bmatrix}_r + \cdots + (-1)^{n-1} \begin{bmatrix} n \\ n-1 \end{bmatrix}_r + (-1)^n \begin{bmatrix} n \\ n \end{bmatrix}_r \\
&= (-1)^r (n-1) \begin{bmatrix} n-1 \\ r \end{bmatrix}_r + (-1)^{r+1} \left(\begin{bmatrix} n-1 \\ r \end{bmatrix}_r + (n-1) \begin{bmatrix} n-1 \\ r+1 \end{bmatrix}_r \right) + \cdots \\
&+ (-1)^{n-1} \left(\begin{bmatrix} n-1 \\ n-2 \end{bmatrix}_r + (n-1) \begin{bmatrix} n-1 \\ n-1 \end{bmatrix}_r \right) + (-1)^n \begin{bmatrix} n-1 \\ n-1 \end{bmatrix}_r \\
&= (n-2) \sum_{k=r}^{n-1} (-1)^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_r \\
&= (n-2)(-1)^r \frac{(n-1-2)!}{(r-2)!} \\
&= (-1)^r \frac{(n-2)!}{(r-2)!}.
\end{aligned}$$

□

For $r = 2$, identity (3.1) simplifies to an elegant form

$$\sum_{k=2}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_2 = (n-2)!.$$

Identity (3.1) simplifies to an (almost equally) elegant form for $r = 3$.

Example 3.2. For $r = 4$ and $n = 7$, according to Theorem 3.1 we have following alternating summation

$$\begin{aligned}\sum_{k=4}^{7}(-1)^k \begin{bmatrix}7\\k\end{bmatrix}_4 &= 120 - 74 + 15 - 1\\ &= 60\left(= (-1)^4\frac{(7-2)!}{(4-2)!}\right).\end{aligned}$$

4 Concluding remarks

The r-Stirling numbers of the first kind are a natural generalization of the Stirling numbers of the first kind, having a similar combinatorial interpretation. The r-Stirling numbers of the first kind are characterised by the requirement that elements 1 through r are in different cycles of a permutation.

Stirling numbers of the first kind, which have the value of the lower index either small or close to the value of the upper index, have known explicit formulas. In this paper, we aimed to find an explicit expression for the r-Stirling numbers as well. It remains to derive other explicit expressions for these numbers. In particular, it would be interesting to answer the question on the recursive nature of such formulas.

Being motivated by the involution which gives the alternating sum of Stirling numbers of the first kind, we consider the analogous problem for the r-Stirling numbers. We presented a family of identities with alternating sign for the rows of triangular matrices of r-Stirling numbers of the first kind. It would be valuable to find a combinatorial proof of these identities. Furthermore, one could find other summations involving these numbers or extend the problem to other generalizations as well as to r-Stirling numbers of the second kind.

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Divisions of narrow strips in square and hexagonal lattices

Tomislav Došlić and Luka Podrug

Abstract

In a recent paper, it was determined that the total number of divisions of a narrow hexagonal strip is counted by odd-indexed Fibonacci numbers. In this paper, we consider two division problems on narrow strips of square and hexagonal lattices. In both cases, we compute the bivariate enumerating sequences and the corresponding generating functions, which allowed us to determine the asymptotic behavior of the total number of such subdivisions and the expected number of parts. For the square case, we extend the results of two recent references by establishing the polynomiality of enumerating sequences forming columns and diagonals of the triangular enumerating sequence. In the hexagonal case, we provide an alternative proof for the number of divisions. We also show how both cases could be treated via the transfer matrix method and discuss some directions for future research.

1 Introduction

It has been a long-standing problem of great practical importance to count the ways of dividing a collection of entities into smaller sets according to a given set of rules. If the entities are considered to be indivisible, and we only care about their number, the natural framework for modeling such situations is the theory of integer partitions and compositions, depending on further properties of the considered entities. If, on the other hand, we are interested in relationships between the entities, such as e.g., their adjacency

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patterns or their relative positions, we must resort to more complex models such as graphs and geometric figures.
In this paper, we look at finite portions of the square and hexagonal regular lattices and count ways of dividing narrow strips in such lattices into a given number of pieces while preserving the integrity of individual squares or hexagons. The considered portions of square and hexagonal lattices remind us of chocolate bars and honeycomb slabs, respectively, hence the title. We start by revisiting some partial results for narrow strips in the square lattice available in the literature and present a complete solution to the problem. In particular, we derive the recurrences satisfied by the sequences enumerating the divisions of a $2 \times n$ strip into k pieces. From them, we compute the bivariate generating function whose univariate specialization yields the recurrence for the overall number of divisions. In that way, we recover the results of Knopfmacher obtained in the context of compositions of ladder graphs [5]. We refine those results by investigating the behavior of columns in the enumerating triangle. We establish convolution-type recurrences for all columns, going thus beyond partial results of references [2, 4, 7]. Then we apply the same approach to narrow strips of hexagons, again deriving the recurrences and computing the bivariate generating function. Then we show how the results for honeycomb strips can be obtained by using transfer matrices. Finally, we also derive transfer matrices for the chocolate bars from which we started.
The paper is concluded with some remarks on the strong and weak points of employed methods and with some indications of possible further directions.

2 Definitions and preliminary results

Let n be a non-negative integer. We consider a $2 \times n$ rectangular strip consisting of $2n$ squares arranged in 2 rows and n columns, such as the one shown in Fig. 1. In the rest of the paper, we will often refer to such strips as chocolate bars of length n. We consider divisions of such structures into a given number of pieces obtained by cutting along the edges of basic squares. More precisely, we would like to find the number of all possible divisions of such a bar of a given length, and also the number of such divisions into a given number of parts k. Clearly, $1 \leq k \leq 2n$ are the only meaningful values of k. Let $r_k(n)$ denote the number of divisions of $2 \times n$ rectangular strip into exactly k pieces and $r(n)$ the total number of divisions. From definition we have that $r_k(n) = 0$ for $k < 1$ and for $k > 2n$. The initial values are $r_1(1) = r_2(1) = 1$, and $r_1(2) = 1$, $r_2(2) = 6$, $r_3(2) = 4$ and $r_1(4) = 1$.
In a recent paper, Brown [2] studied such divisions and obtained a system of recursive relations that we include below as Theorem 2.1. In order to state

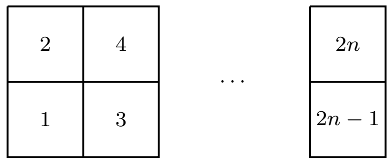

Figure 1: Rectangular strip containing $2n$ squares.

Brown's results, we need one auxiliary term, more specifically, the number of divisions of a $2 \times n$ rectangular strip into k parts such that the squares in the last column belong to different parts. We denote that number by $q_k(n)$ and show one such division in Figure 2 as an example.

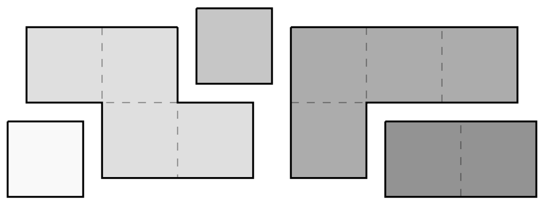

Figure 2: One division of 2×6 rectangular strip into 5 parts with squares in the last column being in the different parts. The total number of such divisions is denoted by $q_5(6)$.

Theorem 2.1 (Brown)**.** *The number of divisions of* $2 \times n$ *strip into* k *parts satisfies following system of equations:*

$$\begin{aligned} r_k(n+1) &= r_k(n) + 3r_{k-1}(n) + r_{k-2}(n) + 2q_k(n) \\ q_k(n+1) &= 2r_{k-1}(n) + r_{k-2}(n) + q_k(n). \end{aligned}$$

It is an easy exercise to eliminate $q_k(n)$ from the system of equations in Theorem 2.1 and to obtain recursive relations for $r_k(n)$,

$$r_k(n+1) = r_{k-2}(n) + 3r_{k-1}(n) + 2r_k(n) + r_{k-2}(n-1) + r_{k-1}(n-1) - r_k(n-1), \tag{2.1}$$

and for the overall number of such divisions,

$$r(n+1) = 6r(n) + r(n-1). \tag{2.2}$$

These recurrences will serve as the starting point of our Section 3, where Brown's results will be extended and refined by establishing recurrences in n for a fixed k and by computing the expected values of k in a random division of a $2 \times n$ chocolate bar.

Next, we consider a hexagonal strip composed of n regular hexagons as shown in Figure 3. Throughout the paper, such hexagonal strips will also be referred to as honeycomb strips. The number to divide hexagonal strips with n hexagons into exactly k parts is given by $\binom{n+k-2}{n-k}$ while the overall number of divisions is F_{2n-1}, where F_n stands for the nth Fibonacci number [3]. Here we present recurrences, generating functions, and again, only divisions along the edges of hexagons are considered. The hexagons are added in the order as it is shown in Figure 3.

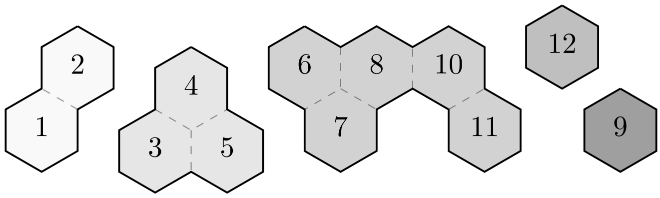

Figure 3: Honeycomb strip with 12 hexagons divided into 5 pieces.

Let $D_k(n)$ denote the set of all possible divisions of the honeycomb strip with n hexagons into k pieces and $d_k(n) = |D_k(n)|$ the number of elements of the set $D_k(n)$. Now we can state some simple cases: $d_1(n) = 1$, for every non-negative integer n, since there is only one way to obtain one part, and $d_n(n) = 1$, since there is only one way to obtain n parts, that is to let each hexagon form its own part. Furthermore, $d_k(n) = 0$ for $k < 1$ and for $k > n$. It is convenient to set $d_1(0) = 1$. As an example, we list all possible divisions of the strip containing 4 hexagons as the first non-trivial case.

$$\begin{aligned}
&d_1(4) = 1 \quad \{1234\} \\
&d_2(4) = 6 \quad \{1, 234\}, \{2, 134\}, \{3, 124\}, \{4, 123\}, \{12, 34\}, \{13, 24\} \\
&d_3(4) = 5 \quad \{12, 3, 4\}, \{13, 2, 4\}, \{23, 1, 4\}, \{24, 1, 3\}, \{34, 1, 2\} \\
&d_4(4) = 1 \quad \{1, 2, 3, 4\}
\end{aligned}$$

Note that the division $\{14, 23\}$ is not included, since hexagons 1 and 4 are not adjacent, as shown in Figure 4, thus cannot form a part.
Since the inner dual of a $2 \times n$ rectangular strip is a subgraph of the inner dual of a hexagonal strip of length $2n$, all divisions of a $2 \times n$ rectangular strip are also valid divisions of a hexagonal strip with $2n$ hexagons, but not vice versa. Figure 5 shows the division $\{1, 23, 4\}$ which is legal in the hexagonal strip but illegal in the rectangular strip.

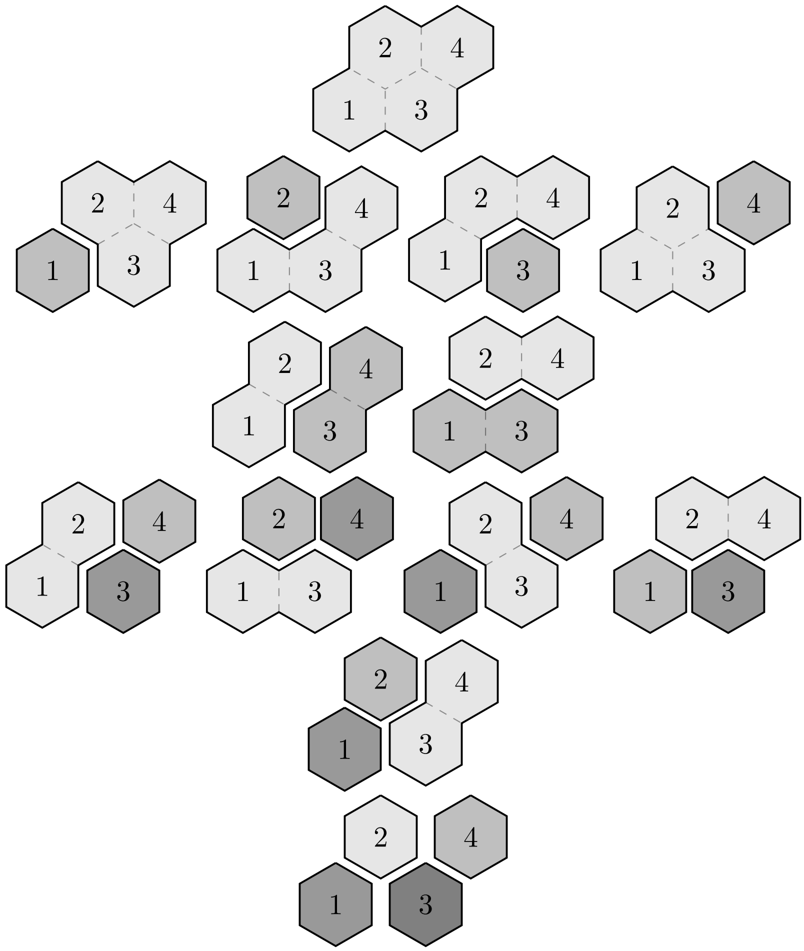

Figure 4: Divisions of the strip with 4 hexagons.

3 Dividing a chocolate bar into a given number of parts

Numbers $r_k(n)$ of (2.1) form a triangular array; its first few lines are shown in Table 3. In this section, we investigate the behavior of its columns, i.e., we turn our attention to recursive relation for $r_k(n)$ where k is fixed.
As mentioned before, $r_0(n) = 0$ and $r_1(n) = 1$, so we look at the first non-trivial case, $k = 2$. From relation (2.1) we obtain $r_2(n) = r_0(n-1) + 3r_1(n-1) + 2r_2(n-1) + r_0(n-2) + r_1(n-2) - r_2(n-2)$. By plugging in

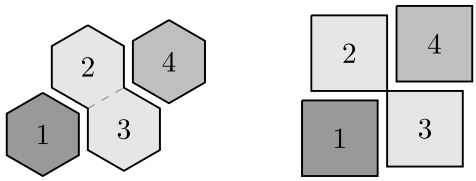

Figure 5: A valid division of a honeycomb strip on the left, and the corresponding division of a rectangular grid that is not allowed on the right side.

$n \backslash k$	1	2	3	4	5	6	7	8	9	10
1	1	1								
2	1	6	4	1						
3	1	15	29	21	7	1				
4	1	28	107	153	111	45	10	1		
5	1	45	286	678	831	603	274	78	13	1

Table 3: First few rows of $r_k(n)$.

$r_0(n-1) = r_0(n-2) = 0$ and $r_1(n-1) = r_1(n-2) = 0$, we obtain

$$\begin{aligned} r_2(n) =& 2r_2(n-1) - r_2(n-2) + 4 \\ r_2(n-1) =& 2r_2(n-2) - r_2(n-3) + 4, \end{aligned}$$

and by subtracting these two equations, we arrive at

$$r_2(n) = 3r_2(n-1) - 3r_2(n-2) + r_2(n-3). \tag{3.1}$$

Rewriting the trivial case

$$r_1(n) = r_1(n-1) \tag{3.2}$$

as

$$\binom{1}{0} r_1(n) = \binom{1}{1} r_2(n-1), \tag{3.3}$$

and case $k = 2$ as

$$\binom{3}{0} r_2(n) = \binom{3}{1} r_2(n) - \binom{3}{2} r_2(n-1) + \binom{3}{3} r_2(n-2) \tag{3.4}$$

suggests that there is a pattern valid also for higher values of k. The conjectured pattern is readily verified by induction, thus yielding the following theorem.

Theorem 3.1. *For integers $n, k \geq 1$ we have*

$$\sum_{j=0}^{2k-1} (-1)^j \binom{2k-1}{j} r_k(n-j) = 0. \tag{3.5}$$

Proof. The proof is by induction. For $k = 1, 2$, the base of induction is true, as stated above. To verify the step of induction, we use recursion (2.1) to obtain a system of $2k-2$ equations as follows:

$$\begin{aligned}
r_k(n) =& r_{k-2}(n-1) + 3r_{k-1}(n-1) + 2r_k(n-1) + \\
&+ r_{k-2}(n-2) + r_{k-1}(n-2) - r_k(n-2) \\
r_k(n-1) =& r_{k-2}(n-2) + 3r_{k-1}(n-2) + 2r_k(n-2) + \\
&+ r_{k-2}(n-3) + r_{k-1}(n-3) - r_k(n-3) \\
r_k(n-2) =& r_{k-2}(n-3) + 3r_{k-1}(n-3) + 2r_k(n-3) + \\
&+ r_{k-2}(n-4) + r_{k-1}(n-4) - r_k(n-4) \\
&\vdots \\
r_k(n-2k+3) =& r_{k-2}(n-2k+2) + 3r_{k-1}(n-2k+2) + 2r_k(n-2k+2) + \\
&+ r_{k-2}(n-2k+1) + r_{k-1}(n-2k+1) - r_k(n-2k+1)
\end{aligned}$$

The term $r_k(n-j)$ appears in at most three equations, namely in the $(j-1)^{\text{st}}$, j^{th} and $(j+1)^{\text{st}}$ equation. To proceed forward, we multiply j-th equation by $(-1)^j \binom{2k-3}{j-1}$ and we add up all equations. For even j, the term $r_k(n-j)$ appears with the coefficient

$$\binom{2k-3}{j-2} + 2\binom{2k-3}{j-1} + \binom{2k-3}{j} = \binom{2k-1}{j},$$

and for odd j with the same coefficient, but with the opposite sign. We conclude that

$$r_k(n) = \sum_{j=1}^{2k-1} (-1)^j \binom{2k-1}{j} r_k(n-j) + A_{k-1}(n) + A_{k-2}(n), \tag{3.6}$$

where $A_{k-1}(n)$ and $A_{k-2}(n)$ are some expressions involving $r_{k-1}(n-j)$ and $r_{k-2}(n-j)$, respectively. The claim of the theorem will be established if we show that both $A_{k-1}(n)$ and $A_{k-2}(n)$ are equal to zero. We first look at $A_{k-1}(n)$. For $j \geq 1$, the term $r_{k-1}(n-j)$ appears twice in our system of equations, in the $(j-1)^{\text{st}}$ and in the j^{th} equation, hence, the coefficient by $r_{k-1}(n-j)$ is $\binom{2k-3}{j-2} - 3\binom{2k-3}{j-1}$ for an odd j, and $3\binom{2k-3}{j-1} - \binom{2k-3}{j-2}$ for an even j. So,

$$A_{k-1}(n) = \sum_{j=1}^{2k-1} (-1)^j \left(3\binom{2k-3}{j-1} - \binom{2k-3}{j-2} \right) r_{k-1}(n-j).$$

For $k-1$ we can use the induction hypothesis, hence

$$3\sum_{j=0}^{2k-3}(-1)^j\binom{2k-3}{j}r_{k-1}(n-j-1)=0$$

and

$$\sum_{j=0}^{2k-3}(-1)^j\binom{2k-3}{j}r_{k-1}(n-j-2)=0.$$

After adding the equations, we have

$$\begin{aligned}
0&=3\sum_{j=0}^{2k-3}(-1)^j\binom{2k-3}{j}r_k(n-j-1)+\sum_{j=0}^{2k-3}(-1)^j\binom{2k-3}{j}r_k(n-j-2)\\
&=3\sum_{j=1}^{2k-2}(-1)^{j-1}\binom{2k-3}{j-1}r_k(n-j)+\sum_{j=2}^{2k-1}(-1)^j\binom{2k-3}{j-2}r_k(n-j)\\
&=\sum_{j=1}^{2k-1}(-1)^j\left(\binom{2k-3}{j-2}-3\binom{2k-3}{j-1}\right)r_k(n-j),
\end{aligned}$$

hence, $A_{k-1}(n)=0$.
Similarly, $A_{k-2}(n)$ can be expressed as

$$A_{k-2}(n)=\sum_{j=1}^{2k-1}(-1)^j\left(\binom{2k-3}{j-1}-\binom{2k-3}{j-2}\right)r_{k-2}(n-j),$$

and, again, by using the induction hypothesis for $k-2$, we obtain $A_{k-2}(n)=0$. The proof follows along the same lines as for $A_{k-1}(n)=0$, and we omit the details.
This completes our proof. □

Theorem 3.1 implies that all columns of the array $r_k(n)$ are polynomials in n. Moreover, $r_k(n)$ is a polynomial in n of degree $2k-2$. The exact expressions can be easily obtained by fitting to the initial values, but we omit the details. Our Theorem 3.1 reestablishes the polynomiality results of references [2] and [4] in a more compact and self-contained form.
A similar reasoning could also be employed near the upper end of the range of k and used to establish polynomiality of diagonals $r_{2n-k}(n)$, going thus beyond the results of references [2, 4]. Indeed, $r_{2n}(n)=1$ for all non-negative integers n. Furthermore, $r_{2n-1}(n)=3n-2$, since among the $2n-1$ pieces there must be exactly one dimer. That dimer is an edge in the inner dual of our bar, hence an edge in a ladder graph with n rungs, and there are exactly $3n-2$ such edges. In a similar way, one can see that

$r_{2n-2}(n)$ must be a quadratic polynomial in n: A division into $2n-2$ parts can either contain one trimer and $2n-3$ monomers, or two dimers and $2n-4$ monomers. As the number of trimers is linear in n and the number of pairs of dimers is quadratic in $3n$, by fitting on the first few values for small n one obtains $r_{2n-2}(n) = \frac{9}{2}(n-1)(n-\frac{2}{3})$. By continuing with the same reasoning, one obtains a general result.

Theorem 3.2. *$r_{2n-k}(n)$ is a polynomial of degree k in n with the leading coefficient $\frac{3^k}{k!}$.*

We leave the details to the interested reader.
We now move towards computing the bivariate generating function for $r_k(n)$. Let

$$F(x,y) = \sum_{n\geq 1}\sum_{k\geq 1} r_k(n)x^n y^k$$

denote the desired generating function. By starting from recurrence (2.1) we readily obtain

$$F(x,y) = \frac{xy(1-x+y+xy)}{1-(2+3y+y^2)x-(y^2+y-1)x^2}.$$

By substituting $y=1$ we obtain

$$F(x,1) = \frac{2x}{1-6x-x^2},$$

the univariate generating function for the sequence r_n.
Now we can determine the expected number of pieces in a random division. We rely on the following version of Darboux's theorem [1].

Theorem 3.3 (Darboux)**.** *If the generating function $f(x) = \sum_{n\geq 0} a_x^n$ of a sequence (a_n) can we written in the form $f(x) = \left(1-\frac{x}{\omega}\right)^{\alpha} h(x)$, where ω is the smallest modulus singularity of f and h is analytic in ω, then $a_n \sim \frac{h(\omega)n^{-\alpha-1}}{\Gamma(-\alpha)\omega^n}$, where Γ denotes the gamma function.*

Since $\omega = \sqrt{10}-3$ we can write

$$F(x,1) = \frac{2x}{x\left(\sqrt{10}-3\right)+1}\left(1-\frac{x}{\left(\sqrt{10}-3\right)}\right)^{-1}.$$

Hence, we have $h(x) = \dfrac{2x}{x\left(\sqrt{10}-3\right)+1}$ and $h(\omega) = \frac{\sqrt{10}}{10}$. Furthermore,

$$\left.\frac{\partial F(x,y)}{\partial y}\right|_{y=1} = \frac{-x(x^3+3x^2+7x-3)}{\left(x\left(\sqrt{10}-3\right)x+1\right)^2}\left(1-\frac{x}{\left(\sqrt{10}-3\right)}\right)^{-2}$$

yields $g(x) = \frac{-x(x^3+3x^2+7x-3)}{\left(x(\sqrt{10}-3)x+1\right)^2}$ and $g(\omega) = \frac{3\sqrt{10}-4}{20}$. By Theorem 3.3, the expected number of parts is

$$\frac{\frac{g(\omega)n}{\Gamma(2)\omega^n}}{\frac{h(\omega)}{\Gamma(1)\omega^n}} = \left(\frac{3}{2} - \sqrt{\frac{2}{5}}\right) n.$$

Hence, we have established the following result for the expected number of parts in a random division of a chocolate bar of length n.

Theorem 3.4. *The expected number of parts in a random division of a chocolate bar of length n is given by*

$$\left(\frac{3}{2} - \sqrt{\frac{2}{5}}\right) n \approx 0.867544\, n.$$

The above result is derived under the so-called equilibrium assumption, where all divisions are equally likely.
The triangle of Table 3 is not (yet) in the OEIS [6]. However, its row sums appear as A078469, the number of compositions of ladder graphs in the sense of reference [5]. Hence, our results could also be interpreted as a refinement of the number of compositions of ladder graphs. Sequence $r_3(n)$ appears as A345897, with the same interpretation as we give here. Curiously, such an interpretation seems to be missing among many combinatorial interpretations of A000384, the hexagonal numbers, which appear as the second column of our triangle. Similarly, $r_{2n-2}(n)$ appears as A081266, but without the interpretation given here.

4 Divisions of honeycomb strips

4.1 Recurrences, explicit formulas and generating functions

Recall that $D_k(n)$ denotes the set of all possible divisions of the honeycomb strip with n hexagons into k pieces, and $d_k(n) = |D_k(n)|$ is the number of elements of the set $D_k(n)$. In order to count the divisions correctly, special attention must be paid to the rightmost two cells, since the new $(n+1)^{\text{st}}$ cell can interact only with them. Whether these hexagons are in the same pieces or not plays a crucial role in how the new hexagon can be added. We denote by $S_k(n)$ the set of all possible divisions of the honeycomb strip with n hexagons into k pieces, with the last two hexagons in the different parts. Similarly, let $T_k(n)$ denote the set of all possible divisions of the strip into k pieces with the two rightmost hexagons belonging to the same piece. Let

$s_k(n) = |S_k(n)|$ and $t_k(n) = |T_k(n)|$. Since the last two hexagons can either be together or separated, we have divided the set $D_k(n)$ into two disjoint sets, $D_k(n) = S_k(n) \cup T_k(n)$, hence $d_k(n) = t_k(n) + s_k(n)$.

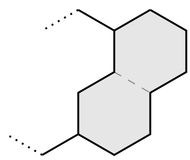

Figure 6: A honeycomb strip with two rightmost hexagons in the same piece.

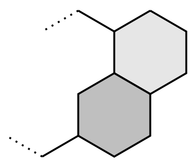

Figure 7: A honeycomb strip with two rightmost hexagons in different pieces.

We first establish an auxiliary result.

Theorem 4.1. *For $n \geq 1$, the number of all possible divisions $s_k(n)$ of the honeycomb strip with n hexagons into k pieces, with hexagons in the last column being in the different pieces, satisfies the following relation:*

$$s_k(n+1) = s_{k-1}(n) + 2s_k(n) - s_k(n-1). \tag{4.1}$$

Proof. We start with a strip containing n hexagons and add one new hexagon to obtain a strip with $n+1$ hexagons. The new hexagon can either increase the number of parts in the division by 1 or not increase this number. To obtain a division with k pieces, we can only start with the division with $k-1$ or k pieces. These are two disjoint sets, so the number of all divisions will be the sum of these cases.

When starting with division consisting of $k-1$ pieces, we can obtain k pieces by adding the new hexagons as individual pieces. Since there is only one way to do that, the number of divisions that can be obtained this way is $d_{k-1}(n)$. Note that the condition that the rightmost two hexagons belong to different pieces is satisfied, as shown in Figure 8.

It remains to consider one last case. We start with a strip divided into k pieces and we add $(n+1)^{\text{st}}$ hexagon. If the last two hexagons in the division are together, we cannot add new hexagons so that the number of parts remains the same, and the new hexagons are in different pieces.

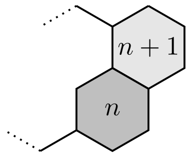

Figure 8: The element of $S_k(n+1)$ obtained from the element of $D_{k-1}(n)$ by adding the new hexagon as separated piece.

Now we move to the case where the last two hexagons in the division are separated. There is only one way to add new hexagons to the existing strip, to put the $(n+1)^{\text{st}}$ hexagon together with $(n-1)^{\text{st}}$ (see Figure 9). Every other layout would be in contradiction with either the number of pieces or the fact that the last two hexagons should be separated, since putting $(n+1)^{\text{st}}$ hexagon together with n^{th} hexagon would produce the element of $T_k(n)$. So in this case, we have $s_k(n)$ ways to obtain the desired division.

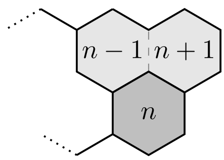

Figure 9: The element of $S_k(n+1)$ obtained from the element of $S_k(n)$ by joining the new hexagon with $(n-1)^{\text{st}}$ hexagon

By summing these two cases, we obtain the recursive relation

$$s_k(n+1) = d_{k-1}(n) + s_k(n). \tag{4.2}$$

To eliminate $d_{k-1}(n)$ from relation (4.2), we use the fact that $d_{k-1}(n) = t_{k-1}(n)+s_{k-1}(n)$. By removing the last hexagon from the strip, we establish a 1-to-1 correspondence between all divisions of a strip with $n-1$ hexagons and divisions of a strip with n hexagons where the last two hexagons are in the same part. Hence, $t_k(n) = d_k(n-1)$. Then we have

$$\begin{aligned} s_k(n+1) &= s_{k-1}(n) + t_{k-1}(n) + s_k(n) \\ &= s_{k-1}(n) + d_{k-1}(n-1) + s_k(n), \end{aligned}$$

hence $d_{k-1}(n-1) = s_k(n+1) - s_{k-1}(n) - s_k(n)$, which combined with relation (4.2) yields

$$s_k(n+1) = s_{k-1}(n) + 2s_k(n) - s_k(n-1)$$

and we proved the theorem. $\square$

By disregarding values of k in recursive relation 4.1 we obtain

$$s(n+1) = 3s(n) - s(n-1),$$

where $s(n)$ represents the number of all divisions of a honeycomb strip of length n with the last two hexagons in different parts. Since we obtained the same recursive relation as for bisection of Fibonacci sequence with $s(1) = 0$ and $s(2) = 1$, we have

$$s(n) = F_{2n-2}.$$

Our main result of this section now follows by much the same reasoning as $d_k(n)$ satisfies the same recurrence as $s_k(n)$. We state it without proof.

Theorem 4.2. *For $n \geq 1$, the number of all possible divisions $d_k(n)$ of n honeycomb strip into k pieces satisfies the following relation:*

$$d_k(n+1) = d_{k-1}(n) + 2d_k(n) - d_k(n-1). \tag{4.3}$$

Again, by grouping terms of recurrence 4.3 with respect to n, we obtain the recurrence satisfied by the sequence $d(n)$ counting the total number of subdivisions of a honeycomb strip of length n as

$$d(n+1) = 3d(n) - d(n-1).$$

Taking into account the initial conditions $d(1) = 1$ and $d(2) = 2$ yields a very simple answer.

Theorem 4.3. *The total number of divisions of a honeycomb strip of length n is given by $d(n) = F_{2n-1}$, where F_n denotes the n^{th} Fibonacci number.*

The above theorem yields a nice combinatorial interpretation of the odd-indexed Fibonacci numbers.
With the above result at hand, it is not too difficult to guess the explicit formulas for $d_k(n)$ and $s_k(n)$. The following theorem is easily proved by simply verifying that the proposed expressions satisfy the respective recurrences and initial conditions.

Theorem 4.4. *The number of all divisions $d_k(n)$ of the honeycomb strip with n hexagons into exactly k pieces is*

$$d_k(n) = \binom{n+k-2}{n-k}.$$

The number $s_k(n)$ of all divisions of the honeycomb strip with n hexagons into k pieces such that the two rightmost hexagons belong to different pieces is equal to zero if $n = 1$ and for $n \geq 2$ it is given as

$$s_k(n) = \binom{n+k-3}{n-k}.$$

Even though sequences $r_k(n)$ and $d_k(n)$ satisfy different recursive relations and describe different problems, it turns out that their columns satisfy the same recurrences. Our next theorem is analogous to Theorem 3.1, but for the sequence $d_k(n)$. We state it without proof.

Theorem 4.5. *For $n, k \geq 1$ we have*

$$\sum_{j=0}^{2k-1} (-1)^j \binom{2k-1}{j} d_k(n-j) = 0. \tag{4.4}$$

As with a rectangular strip, we are now interested in the generating function of the sequence $d_k(n)$. Let

$$G(x,y) = \sum_{n\geq 1} \sum_{k\geq 1} d_k(n) x^n y^k.$$

By recursive relation 4.3 we have

$$\begin{aligned} G(x,y) &= xy + x^2 y\,(1+y) + \\ &\quad + \sum_{n\geq 3} \sum_{k\geq 1} \left(d_{k-1}(n-1) + 2d_k(n-1) - d_k(n-2)\right) x^n y^k \\ &= xy + x^2 y\,(1+y) + xy\,(G(x,y) - xy) + \\ &\quad + 2x\,(G(x,y) - xy) - x^2 G(x,y), \end{aligned}$$

so we have

$$G(x,y) = \frac{xy(1 + x(y-1) - xy)}{1 - (2+y)x + x^2}.$$

By putting $y = 1$, we obtain the univariate generating function for the sequence $d(n)$ as

$$G(x,1) = \frac{x - x^2}{1 - 3x + x^2}.$$

Its smallest-modulus singularity is $\omega = \frac{1}{2}\left(3 - \sqrt{5}\right)$ and this gives us the asymptotics of the expected number of pieces in a random divisions of honeycomb strips of a given length.

Theorem 4.6. *The expected number of pieces in a random division of a honeycomb strip of length n asymptotically behaves as*

$$\frac{\sqrt{5}}{5} n \approx 0.447214n.$$

The proof follows by a straightforward application of the Darboux theorem, and we omit the details.

4.2 Some consequences

Our results make it possible to give a new combinatorial interpretation for some famous identities. We present two such cases.
First, by double-counting the set $D(n)$, we gave new meaning to the well-known identity

$$\sum_{k=1}^{n} \binom{n+k-2}{n-k} = F_{2n-1}.$$

Another identity will be proven in the next theorem.

Theorem 4.7. *For $n, m \geq 1$ we have*

$$F_{2n+2m-1} = F_{2n-1}F_{2m-1} + F_{2n}F_{2m}.$$

Proof. We start with two honeycomb strips of lengths n and m. To prove the statement of a theorem, we glue strips together as in Figure 10 and double-count the number of divisions. On one hand, we have a strip of length $n+m$ whose number of divisions is $d(n+m)$. On the other hand, we consider what can happen when strips are glued together. In the first case, parts of each division do not interact, hence, we have $d(n)d(m)$ such divisions. In the other cases, at least two parts, one from each strip, must merge. But to correctly count the number of divisions in those cases, it is important to know whether the division is with the two last hexagons together or separated. If both strips have the last two hexagons together, the total number of such divisions is $t(n)t(m)$. If both strips have the last two hexagons separated, the total number of such divisions is $4s(n)s(m)$, since there are four different ways to merge parts. Finally, if one strip has two last hexagons separated and the other one together, we can merge the parts in two ways. Since either one of the strips can be in both situations, the total number of divisions in this case is $2s(n)t(m) + 2s(n)t(m)$.

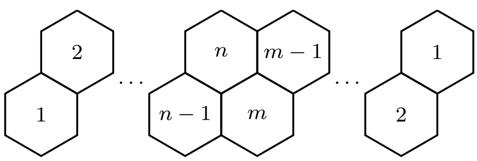

Figure 10: Two honeycomb strips glued together.

So,

$$d(n+m) = d(n)d(m) + 4s(n)s(m) + t(n)t(m) + 4s(n)t(m).$$

The claim now follows by substituting $d(n) = F_{2n-1}$, $s(n) = F_{2n-2}$ and $t(n) = F_{2n-3}$ and rearranging the resulting expressions. $\square$

The results of this section can also be formulated in terms of graph compositions, this time of the graph P_n^2 obtained by adding edges between all pairs of vertices at distance 2 in P_n, the path on n vertices. The following results are a direct consequence of the fact that P_n^2 is the inner dual of a honeycomb strip of length n.

Theorem 4.8. *The number of compositions of P_n^2 with k components is equal to $\binom{n+k-2}{n-2}$. The total number of compositions of P_n^2 is equal to F_{2n-1}.*

5 Transfer matrix method

5.1 Honeycomb strips

In this section, we present another approach to obtain an overall number of divisions, the one based on transfer matrices. It might seem less natural than recurrence relations, but it often turns out to be suitable when recurrence relations are complicated or unknown.

We again consider a honeycomb strip such as the one shown in the Figure 3, and look at its rightmost column, i.e., at the hexagons labeled by $n-1$ and n. There are two possible situations regarding these hexagons: they can be in the same piece of a subdivision, or they can belong to two different pieces. We denote a strip with the last two hexagons together as a type T strip and a strip with the last two hexagons separated as a type S strip. Adding the $(n+1)-$st hexagon might result again in a type S strip or a type T strip. There are altogether four possibilities, each of which produces certain effects on the number of pieces in the resulting strip. For example, if we start with a strip of type S and we want to end with a strip of type S, we can either add the new hexagon to the part which contains the $(n-1)^{\text{st}}$ hexagon, or we can let the $(n+1)^{\text{st}}$ hexagon to form its own part. In the first case, the number of parts will remain the same; in the second case, it will increase by one. Figure 11 shows this case. The main idea of the transfer matrix

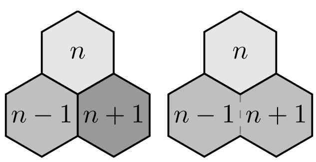

Figure 11: Both cases resulting in a strip of type S.

method is to arrange the effects of adding a single hexagon into a 2×2 matrix whose entries will keep track of the number of pieces via a formal variable, say, y. The rows and columns of such a matrix are indexed by possible states, in our case T and S, and the element at the position S, S in our example will be $1+y$. That clearly captures the fact that transfer from S to S results either in the same number of pieces, or the number of pieces increases by one. The other three possible transitions, $T \to T$, $S \to T$ and $T \to S$ are described by matrix elements $1, 1$, and y, respectively. Indeed, it is clear that adding a hexagon to obtain the rightmost column together cannot increase the number of pieces, hence the two ones, and that starting from T and arriving at S is possible only by the last hexagon forming a new piece, hence increasing the number of pieces by one, hence y. If we denote our matrix by H, we can write it as

$$H(y) = \begin{bmatrix} 1 & 1 \\ y & 1+y \end{bmatrix}.$$

By construction, it is clear that adding a new hexagon will be well described by multiplying some vector of states by our matrix $H(y)$, and that repeated addition of hexagons will correspond to multiplication by powers of $H(y)$. It remains to account for the initial conditions.
For the initial value $n = 1$, we have a trivial case, one hexagon forms one part. For $n = 2$ we have two possibilities, hexagons are in the same part or separated. Hence, his case is represented by a vector

$$\overrightarrow{h_2} = x \begin{bmatrix} y \\ y^2 \end{bmatrix}.$$

By introducing another formal variable, say x, to keep track of the length, the above procedure will produce a sequence of bivariate polynomials whose coefficients are our numbers $d_k(n)$. The first few polynomials are shown in Table 4 after the theorem, which summarizes the described procedure.

Theorem 5.1. *The number of divisions of a honeycomb strip of a length n into k parts is the coefficient of $x^n y^k$ in the expression*

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ y & 1+y \end{bmatrix}^{n-2} \begin{bmatrix} y \\ y^2 \end{bmatrix} x^n. \tag{5.1}$$

The coefficients by $x^n y^k$ in expression (5.1) could now be determined by studying the powers of the transfer matrix. By looking at the first few cases,

$$H(y)^2 = \begin{bmatrix} 1+y & 2+y \\ 2y+y^2 & 1+3y+y^2 \end{bmatrix}$$

and

$$H(y)^3 = \begin{bmatrix} 1+3y+y^2 & 3+4y+y^2 \\ 3y+4y^2+y^3 & 1+6y+5y^2+y^3 \end{bmatrix},$$

n	
1	xy
2	$x^2(y+y^2)$
3	$x^3(y+3y^2+y^3)$
4	$x^4(y+6y^2+5y^3+y^4)$
5	$x^5(y+10y^2+15y^3+7y^4+y^5)$
6	$x^6(y+15y^2+35y^3+28y^4+9y^5+y^6)$

Table 4: First few bivariate polynomials from the transfer matrix method.

we could guess the entries in the general case and then verify them by induction. We state the result, omitting the details of the proof.

Lemma 5.2. *Matrix*

$$H(y)^n = \begin{bmatrix} p(n) & s(n) \\ ys(n) & p(n+1) \end{bmatrix}$$

with $p(n) = \sum_{k=1}^{n} \binom{n+k-2}{n-k} y^{k-1}$ *and* $s(n) = \sum_{k=1}^{n} \binom{n+k-1}{n-k} y^{k-1}$.

Lemma 5.2 allows us to simplify the expression (5.1) to have

$$\begin{aligned}
&\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ y & 1+y \end{bmatrix}^{n-2} \begin{bmatrix} y \\ y^2 \end{bmatrix} x^n = \\
&= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} p(n-2) & s(n-2) \\ ys(n-2) & p(n-1) \end{bmatrix} \begin{bmatrix} y \\ y^2 \end{bmatrix} x^n \\
&= \Big(yp(n-2) + ys(n-2) + y^2 s(n-2) + y^2 p(n-1) \Big) x^n \\
&= \left(yp(n-1) + y\left(s(n-2) + yp(n-1) \right) \right) x^n \\
&= p(n) x^n y \\
&= \sum_{k=1}^{n} \binom{n+k-2}{n-k} y^k x^n
\end{aligned}$$

By Theorem 5.1 we have

$$d(n,k) = \binom{n+k-2}{n-k}.$$

Now we turn our attention to the number of all possible divisions, i.e. we wish to determine the number $d(n)$. To do that, we again use a matrix $H(y)$ and Theorem 5.1. By setting $y = 1$ we have $H(1) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} F_1 & F_2 \\ F_2 & F_3 \end{bmatrix}$. Again, the following claim is easily guessed and verified by induction.

Lemma 5.3. $H(1)^n = \begin{bmatrix} F_{2n-1} & F_{2n} \\ F_{2n} & F_{2n+1} \end{bmatrix}$.

Finally, by Lemma 5.3 we can simplify expression (5.1) to have

$$\begin{aligned}
\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} F_{2n-5} & F_{2n-4} \\ F_{2n-4} & F_{2n-3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} F_{2n-5} + F_{2n-4} & F_{2n-4} + F_{2n-3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} F_{2n-3} & F_{2n-2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= F_{2n-1}.
\end{aligned}$$

By Theorem 5.1 we have $d(n) = F_{2n-1}$.

5.2 Chocolate bars

Transfer matrices can also be used to obtain the sequence $r_k(n)$ denoting the number of ways to divide a rectangular strip $2 \times n$ into k parts. In this case, we do not add square by square, but column by column. So, let T denote a division of a strip where squares in the last column are in the same part, and S a division where squares in the last column are in different parts.

For $n = 1$, we have the same case as $n = 2$ in a honeycomb strip, so this case is represented by a vector

$$\overrightarrow{q_1} = x \begin{bmatrix} y \\ y^2 \end{bmatrix}.$$

Similar to the honeycomb case, if we start with a division of type T and we wish to obtain another division of type T, we can do that either by appending two new squares to the same part as the squares of the last column, or we can let two new squares form a new part. Hence, the corresponding entry in the transfer matrix is $1 + y$. By doing a similar analysis for other cases, we obtain the transfer matrix

$$Q(y) = \begin{bmatrix} 1+y & 2+y \\ y(2+y) & (1+y)^2 \end{bmatrix}.$$

Again, y is a formal variable keeping track of the number of pieces. So, for a strip $2 \times n$, the coefficient by $x^n y^k$ in the expression

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1+y & 2+y \\ y(2+y) & (1+y)^2 \end{bmatrix}^{n-1} \begin{bmatrix} y \\ y^2 \end{bmatrix} x^n$$

represents the number of ways to divide a rectangular strip $2 \times n$ into exactly k parts.

We conclude this section by mentioning that in both cases, we could have obtained the asymptotic behavior of numbers $d(n)$ and $r(n)$ by computing the leading eigenvalue of the corresponding transfer matrix.

6 Concluding remarks

In this paper, we have employed two different methods to count divisions of narrow strips of squares and hexagons, respectively, into a given number of pieces, when cutting is allowed only along the edges of basic polygons. We have obtained several triangular integer arrays and determined formulas for their entries. Despite similar settings, the two problems behave in different ways: for honeycomb strips, the entries of the enumerating triangles are given as binomial coefficients with parameters dependent on the strip length and the number of pieces, while for chocolate bars, no closed-form expression has been obtained. We were able to show, though, that the entries in columns satisfy convolution-type recurrences with coefficients forming alternating rows of Pascal triangle.

Both problems were then addressed by using the transfer-matrix formalism. The original results for the total number of divisions were re-derived in a more compact way, demonstrating thus the power of the transfer-matrix method. However, we found the approach unsuitable for refining the aggregate results, for establishing the polynomial nature of columns, and for obtaining closed-form solutions in the rectangular case. Nevertheless, we believe that the transfer matrices would prove useful in treating several similar problems, as indicated by our experiments with wider strips in both square and hexagonal lattices and with narrow strips in the triangular lattice.

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Surprising bijections of the Riviera model

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Abstract

In this short note we give bijective proofs of two results from [3] which establish a correspondence between the maximal configurations in the Riviera model and two seemingly unrelated combinatorial objects: strongly restricted permutations and closed walks on a certain small graph. The results in the original paper were established by inspecting generating functions of the enumerating sequences.

1 Introduction

In [3] we introduced the *Riviera model* as a variant of the combinatorial settlement planning model first studied in [7, 8] and further developed in [2, 4]. The original problem, motivated by a real-life application and analyzed in [7, 8], proved to be too complex for analytical treatment, so we considered a 1-dimensional toy model which we were able to fully solve. Here, we briefly recall the setup. We begin with a finite strip of land divided into consecutive lots of equal size. Each lot can be occupied by a house or left vacant. The first requirement is that each occupied lot must be adjacent to at least one vacant lot. This resembles a Mediterranean settlement along the coast (hence the name 'Riviera model') stretching in the east-west direction, where each house is exposed to sunlight from the south and at least one

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additional side (east or west depending on which side the adjacent vacant lot is). We can concisely write such configurations as binary words with ones denoting occupied and zeros vacant lots. The configurations satisfying this first requirement are called *permissible*. An example of a permissible configuration is:

$$110110001010011.$$

Note that the two houses on the left and right edge of the strip are assumed to be exposed to sunlight from west and east respectively. The second requirement is that the configurations be *maximal*, meaning that no further houses can be added to the vacant lots without violating the first insolation requirement. Clearly, the configuration above is not maximal, but the following is:

$$110110101011011.$$

It is clear that the first (permissibility) requirement is equivalent to asking that no three consecutive ones appear anywhere in the configuration. And it is not hard to see that the second (maximality) requirement can be verified by inspecting substrings of length 4 of the given configuration. To be precise, the following lemma was proven in [3].

Lemma 1.1 (Lemma 2.1 in [3]). *Let $n \in \mathbb{N}$. A configuration in the Riviera model is maximal if and only if, when padded with zeros, it does not contain any of the following (decorated[1]) substrings:*

$$1\underline{1}1, \quad 0\underline{0}0, \quad 01\underline{0}0, \quad 0\underline{0}10. \tag{1.1}$$

Next, with Lemma 1.1 in hand, the standard transfer matrix method approach was employed in [3] to establish a one-to-one correspondence between maximal Riviera configurations and walks on the directed graph in Figure 1. This digraph was created by taking all the allowed (i.e. not forbidden) substrings of length 3 and adding a directed edge from $u_1u_2u_3$ to $v_1v_2v_3$ if and only if they *overlap progressively*, meaning that $u_2 = v_1$ and $u_3 = v_2$; and if the substring $u_1u_2u_3v_3 = u_1v_1v_2v_3$ is not forbidden. The shaded starting/ending nodes were determined based on the boundary condition, which in our case states that the edge lots receive sunlight from the boundary side. By standard methods, see [3], one can now write the transfer matrix associated to this graph and obtain the bivariate generating function of the

[1] When inspecting whether a configuration $c_1 \dots c_n$ contains a decorated word $d_1 \dots \underline{d_k} \dots d_l$, we check against a padded word $\dots 000c_1 \dots c_n000 \dots$ but with the underlined letter of the decorated word aligned with c_i for $i = 1, \dots, n$. This is necessary as e.g. the configuration 10011 would otherwise be considered allowed (not containing any of the forbidden substrings), although it is not maximal.

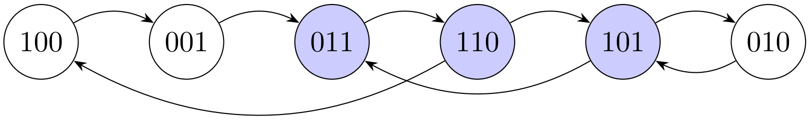


Figure 1: (Figure 3 in [3]) Transfer digraph $\mathcal{G}_{\mathcal{R}}$ for the Riviera model. For example, a maximal configuration 110011010110 is represented by the walk: $110 \to 100 \to 001 \to 011 \to 110 \to 101 \to 010 \to 101 \to 011 \to 110$. Each walk must start and end at shaded nodes.

sequence $(J_{k,n})$ enumerating maximal Riviera configurations of prescribed length n and occupancy k:

$$g(x,y) = \frac{1 + xy - (x - x^2)y^2 + x^2y^3 - x^3y^5}{1 - xy^2 - x^2y^3 - x^2y^4 + x^3y^6} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} J_{k,n}x^k y^n. \qquad (1.2)$$

Remark 1.2. Setting $x = 1$ in the expression (1.2) above, one obtains $g(1,y) = f(y)$, the generating function for the sequence (f_n) which counts the number of all maximal configurations of length n:

$$f(y) = \frac{1 + y + y^3 - y^5}{1 - y^2 - y^3 - y^4 + y^6} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} J_{k,n}\right) y^n = \sum_{n=0}^{\infty} f_n y^n. \qquad (1.3)$$

Similarly, by setting $y = 1$ in (1.2), one obtains $g(x,1) = h(x)$, the generating function for the sequence (h_k) which counts the number of maximal configurations (of variable length) with a fixed number k of occupied lots:

$$h(x) = \frac{1 + 2x^2 - x^3}{1 - x - 2x^2 + x^3} = \sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty} J_{k,n}\right) x^k = \sum_{k=0}^{\infty} h_k x^k. \qquad (1.4)$$

By expanding the bivariate generating function $g = g(x,y)$, we obtain the precise distribution of the occupancies of maximal configurations relative to their length. The first few coefficients in the expansion of $g(x,y)$ are given in Table 5. Note that the ratio $\frac{k}{n}$ for non-zero coefficients stabilizes between $\frac{1}{2}$ and $\frac{2}{3}$ as n becomes large.
The first several values of the sequence (f_n), associated with the generating function $f(y)$, can be obtained as column sums of Table 5:

$$1, 1, 1, 3, 3, 4, 6, 9, 12, 16, 24, 33, 46, 64, \ldots$$

Similarly, the first several values of the sequence (h_k), associated with the generating function $h(x)$, can be obtained as row sums of Table 5:

$$1, 1, 5, 5, 14, 19, 42, 66, 131, \ldots$$

Table 5: The coefficients of $x^k y^n$ in the expansion of the bivariate generating function $g(x, y)$.

$k\backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	1																	
1		1																
2			1	3	1													
3					2	3												
4						1	6	6	1									
5								3	10	6								
6									1	10	20	10	1					
7											4	22	30	10				
8												1	15	49	50	15	1	
9														5	40	91	70	15
10															1	21	100	168

Upon consulting The On-Line Encyclopedia of Integer Sequences (OEIS) [9], we realized that these sequences had been already studied in quite different settings. The sequence (f_n) appears as OEIS sequence A080013 and is described as counting certain strongly restricted permutations. The sequence (h_n) appears as OEIS sequence A096976 and is described as counting closed walks on the path graph P_3 with a loop. The equipotency of the pairs of these combinatorial objects follows directly from the fact that their generating functions match (up to a finite degree polynomial). However, it is possible to construct explicit bijections demonstrating these equipotencies. This is the content of the next two sections.

2 The Riviera model and strongly restricted permutations

The notion of strongly restricted permutations was introduced by Lehmer in [5]. If W is some fixed subset of integers, one would like to count the number of all the permutations $\pi \in S_n$[2] such that $\pi(i) - i \in W$, for all $i \in [n]$.

In [10, Examples 4.7.9, 4.7.17–18] two techniques are presented for obtaining the generating function for the number of strongly restricted permutations for some particular sets W, namely the transfer-matrix method and the technique using factorization in free monoids.

In [1], Baltić devised a new technique for counting restricted permutations in case $\min W = -k$ and $\max W = r$ for some positive integers $k \leq r$. When $W = \{-2, -1, 2\}$, the sequence counting the corresponding restricted permutations of length n appears in OEIS under the number A080013. The

[2] S_n denotes the set of all permutations of the set $[n] = \{1, \ldots, n\}$.

generating function of that sequence is $\dfrac{1-y^2}{1-y^2-y^3-y^4+y^6}$. Note that

$$\begin{aligned}\frac{1-y^2}{1-y^2-y^3-y^4+y^6} &= 1+y^3+y^4\cdot\frac{1+y+y^3-y^5}{1-y^2-y^3-y^4+y^6}\\ &= 1+y^3+y^4\cdot f(y),\end{aligned}$$

where $f(y)$ is the generating function for the number of maximal Riviera configurations of fixed length n, see (1.3). From here, the following result is immediate.

Theorem 2.1. *The number of maximal configurations of length n in the Riviera model is equal to the number of permutations π of length $n+4$, which satisfy the constraint*

$$\pi(i)-i\in\{-2,-1,2\}. \tag{2.1}$$

It turns out that one can construct a natural bijection between these two types of objects. The idea is to encode restricted permutations as walks on some digraph, similar to the one in Figure 1. If those two graphs are isomorphic, this isomorphism would automatically produce a bijection between the underlying combinatorial objects.
To construct this digraph we, once again, use the transfer-matrix method. One can argue as in [10, Example 4.7.9] to show that the method is applicable in this case. Let $\pi\in S_n$ be a permutation for which $\pi(i)-i\in W=\{-2,-1,2\}$, for all $i\in[n]$. One can rewrite such a permutation as a sequence of symbols in W. In order to check that such a sequence $u_1\dots u_n$ corresponds to a valid permutation, it suffices to check all the substrings of length 5. This is because the function $\sigma\colon[n]\to[n]$ defined as $\sigma(i)=i+u_i$ will be a permutation as soon as it is onto; and for this, one only needs to check whether $i\in\{\sigma(i-2),\sigma(i-1),\sigma(i),\sigma(i+1),\sigma(i+2)\}$, for all $3\le i\le n-2$. Additionally, one needs to check that 1 and 2 are in the set $\{\sigma(1),\sigma(2),\sigma(3),\sigma(4)\}$, and that $n-1$ and $n-2$ are in the set $\{\sigma(n-3),\sigma(n-2),\sigma(n-1),\sigma(n)\}$. The effect of this being that the walks must start and end at a certain subset of vertices of the constructed digraph. From here, one can write Algorithm 1 that produces this digraph which is an induced subgraph of the de Bruijn graph over the set of all 5 letter words in the alphabet $\{-2,-1,2\}$.
The digraph $\mathcal{G}$ constructed in Algorithm 1 has the vertex set $\mathcal{V}$ consisting of 30 allowed words of length 5. It turns out that this graph can be further condensed to give a smaller representation of our strongly restricted permutations. If one considers all the 4-letter words $\{-2,-1,2\}^4$ that do not appear as substrings of the 30 allowed words, one gets 59 forbidden words of

Algorithm 1 The creation of the digraph $\mathcal{G}$ for strongly restricted permutations

AllowedNodes $= \emptyset$
StartNodes $= \emptyset$
EndNodes $= \emptyset$
for $u_1u_2u_3u_4u_5 \in \{-2,-1,2\}^5$ **do**
  if $3 \in \{1+u_1, 2+u_2, 3+u_3, 4+u_4, 5+u_5\}$ **then**
    add node $u_1u_2u_3u_4u_5$ to AllowedNodes
    if $1,2 \in \{1+u_1, 2+u_2, 3+u_3, 4+u_4, 5+u_5\}$ **then**
      add node $u_1u_2u_3u_4u_5$ to StartNodes
    end if
    if $4,5 \in \{1+u_1, 2+u_2, 3+u_3, 4+u_4, 5+u_5\}$ **then**
      add node $u_1u_2u_3u_4u_5$ to EndNodes
    end if
  end if
end for

$\mathcal{E} = \emptyset$
for $u_1u_2u_3u_4u_5, v_1v_2v_3v_4v_5 \in$ AllowedNodes **do**
  if $u_2u_3u_4u_5 = v_1v_2v_3v_4$ **then**
    add edge $u_1u_2u_3u_4u_5 \to v_1v_2v_3v_4v_5$ to $\mathcal{E}$
  end if
end for

$\mathcal{V} = \emptyset$
for $u_1u_2u_3u_4u_5 \in$ AllowedNodes **do**
  if there is a path starting in StartNodes, passing through $u_1u_2u_3u_4u_5$ and ending in EndNodes **then**
    add node $u_1u_2u_3u_4u_5$ to $\mathcal{V}$
  end if
end for

remove from $\mathcal{E}$ all the edges not involving nodes in $\mathcal{V}$
return $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

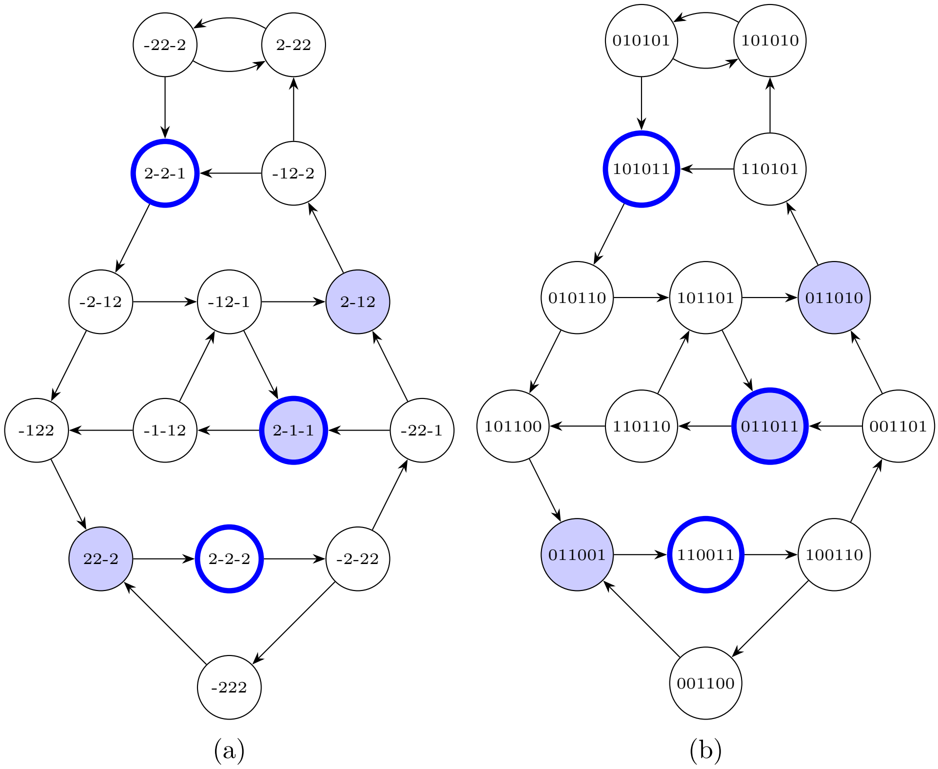


Figure 2: The digraph $\mathcal{G}_{\mathcal{P}}$ in 2(a) encodes strongly restricted permutations satisfying the constraint (2.1). The starting nodes are shaded and thicker outlines indicate the ending nodes. The digraph $\mathcal{G}'_{\mathcal{R}}$ in 2(b) encodes configurations of the Riviera model using substrings of length 6. The nodes corresponding to the highlighted nodes in 2(a) via the unique digraph isomorphism are shaded and outlined in this graph too.

length 4. By inspection, one can check that each of the $213 = 3^5 - 30$ forbidden 5-letter words contains one of the 4-letter forbidden words which means that the same information contained in $\mathcal{G}$ can be encoded in a digraph with a vertex set consisting of only $22 = 3^4 - 59$ 4-letter words. Finally, if we use edges to encode allowed words, rather than just taking the whole induced subgraph of the corresponding de Bruijn graph, we can condense this digraph even further, and obtain the digraph in Figure 2(a) with 15 nodes representing allowed 3-letter words and an edge from $u_1u_2u_3$ to $v_1v_2v_3$ if and only if $u_1u_2u_3v_3 = u_1v_1v_2v_3$ is allowed 4-letter word. The highlighted nodes are either starting or ending nodes, or, in one case, both.
We would now like to match the digraph in Figure 2(a), call it $\mathcal{G}_{\mathcal{P}}$, with the digraph in Figure 1, call it $\mathcal{G}_{\mathcal{R}}$. Unfortunately, they are not isomorphic,

but we can try to create higher edge graphs from the digraph $\mathcal{G}_{\mathcal{R}}$, details below, which encode the same information as $\mathcal{G}_{\mathcal{R}}$ — in hope of obtaining a graph isomorphic to $\mathcal{G}_{\mathcal{P}}$. This process is opposite of 'condensation' we have performed to the digraph produced by the Algorithm 1 in order to obtain the digraph $\mathcal{G}_{\mathcal{P}}$.

We have already noted that the graph $\mathcal{G}_{\mathcal{R}}$ shown in Figure 1 is a subgraph of the 3-dimensional de Bruijn graph over the alphabet $\{0,1\}$. We can construct a subgraph of the n-dimensional de Bruijn digraph, where $n > 3$, over the same alphabet, which will encode the same information as $\mathcal{G}_{\mathcal{R}}$. It turns out that $n = 6$ will do. The vertex set of this, so called, higher edge graph $\mathcal{G}'_{\mathcal{R}}$ consists of all the allowed words of length 6, which one can think of as all the possible walks of length 3 on the graph $\mathcal{G}_{\mathcal{R}}$. A directed edge from $c_1 \dots c_6$ to $d_1 \dots d_6$ is added to the edge set of $\mathcal{G}'_{\mathcal{R}}$ if and only if the corresponding words overlap progressively $(c_2 \dots c_6 = d_1 \dots d_5)$. The digraph $\mathcal{G}'_{\mathcal{R}}$ is, therefore, the vertex-induced subgraph of the corresponding 6-dimensional de Bruijn graph. For more details on construction of higher edge graphs, see [6, Definition 2.3.4].

The graph $\mathcal{G}'_{\mathcal{R}}$ obtained by the above procedure, is shown in Figure 2(b). Note that it is isomorphic to $\mathcal{G}_{\mathcal{P}}$, and that this isomorphism is unique. Also note that the set of nodes at which the walks on $\mathcal{G}'_{\mathcal{R}}$ would be allowed to start and end is much larger than the set highlighted in Figure 2(b). More precisely, any node $c_1 \dots c_6$ for which $c_1c_2c_3 \in \{110, 101, 011\}$ would be a starting node, and if $c_4c_5c_6 \in \{110, 101, 011\}$, it would be an ending node. But the walks of length $n+1$ on $\mathcal{G}'_{\mathcal{R}}$ would then account for all the maximal configurations in the Riviera model of length $n+7$ — and that is not what we want, since the walks of length $n+1$ on $\mathcal{G}_{\mathcal{P}}$ encode the strictly restricted permutations of $[n+4]$.

If we consider the walks on $\mathcal{G}'_{\mathcal{R}}$ which start and end at the nodes that correspond to starting and ending nodes in $\mathcal{G}_{\mathcal{P}}$, we immediately note that all the configurations obtained in such a way always start with 0110 and end with 011. Using the graph $\mathcal{G}_{\mathcal{R}}$ in Figure 1 it is clear that adding the prefix 0110 and suffix 011 to a maximal configuration, again produces a 7-blocks longer (permissible) maximal configuration. This is because from each starting node, there is a backward path (going along edges in the direction opposite to the arrow direction) of length 4 which produces the prefix 0110; also from each ending node, there is a 3-step continuation of path which produces the suffix 011. Conversely, removing that same prefix and suffix from a maximal configuration of length $n+7$, produces a maximal configuration of length n. We can again argue using the graph $\mathcal{G}_{\mathcal{R}}$. Any walk starting with $011 \to 110$ after three steps must again reach one of the starting nodes; and walk ending in 011 when traced backwards must, after three steps going backwards, reach one of the ending nodes. This shows

that there is a bijective correspondence between all the maximal Riviera configurations of length n and the maximal Riviera configurations of length $n+7$ starting with 0110 and ending with 011 which in turn are in a bijective correspondence with the strongly restricted permutations of length $n+4$. The bijection is obtained by translating walks on $\mathcal{G}_{\mathcal{P}}$ to walks on $\mathcal{G}'_{\mathcal{R}}$ and the other way around.
It is, in fact, possible to specify this bijection even more concisely, circumventing the graphs in Figure 2 altogether. Compare each edge in $\mathcal{G}_{\mathcal{R}}$ with all its associated edges in $\mathcal{G}'_{\mathcal{R}}$ and note that the corresponding edges in graph $\mathcal{G}_{\mathcal{P}}$ all represent adding the same symbol at the end. E.g. the transition $011 \to 110$ in $\mathcal{G}_{\mathcal{R}}$ corresponds to transitions $101011 \to 010110$, $011011 \to 110110$ and $110011 \to 100110$ in $\mathcal{G}'_{\mathcal{R}}$ and all of them in $\mathcal{G}_{\mathcal{P}}$ correspond to adding the letter 2 at the end. Collecting all this information together, we can label the edges of the graph $\mathcal{G}_{\mathcal{R}}$ in Figure 1 with the appropriate letter which is being added in the permutation graph $\mathcal{G}_{\mathcal{P}}$ corresponding to that transition. This edge-labeled graph is given in Figure 3.
We now summarize how to bijectively map any maximal Riviera configuration of length n to a strongly restricted permutation of length $n+4$ using Figure 3. Take any such maximal configuration and prefix it with 0110 and suffix it with 011. Then take a walk over the graph in Figure 3 (which will be of length $n+7-3=n+4$) and collect the labels $u_1 \dots u_{n+4}$ of all the edges traversed. Finally, construct the bijection $\sigma\colon [n+4] \to [n+4]$ as $\sigma(i) = i + u_i$ for $i \in [n+4]$.
As an example, the maximal configuration 10110 is first enlarged to the maximal configuration 0110|10110|011. Next, we examine the unique walk determined by this configuration: $011 \xrightarrow{2} 110 \xrightarrow{-1} 101 \xrightarrow{2} 010 \xrightarrow{-2} 101 \xrightarrow{-1} 011 \xrightarrow{2} 110 \xrightarrow{2} 100 \xrightarrow{-2} 001 \xrightarrow{-2} 011$. This walk generates the permutation σ encoded with the string 2-12-2-122-2-2, which is the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 5 & 2 & 4 & 8 & 9 & 6 & 7 \end{pmatrix}.$$

We end this section with a remark which will prove useful in the next section.

Remark 2.2. Above, we have argued that taking any maximal configuration $c_1 \dots c_n$ and prefixing it with 0110 and suffixing it with 011 yields a bijection between all the maximal Riviera configurations of length n and the maximal Riviera configurations of length $n+7$ starting with 0110 and ending with 011.
If we further add prefix 10 and suffix 001 to these already extended configurations, we obtain a bijective correspondence between the maximal Riviera

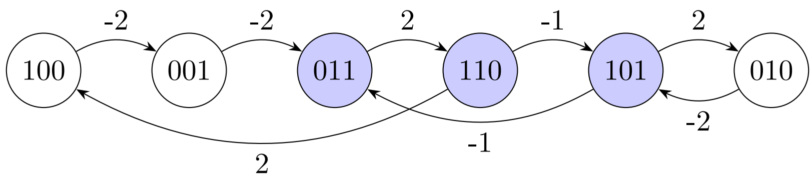


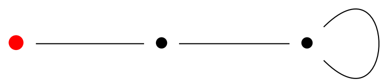
Figure 3: The digraph $\mathcal{G}_{\mathcal{R}}$ with labeled edges which encodes the bijection between maximal configurations of length n in the Riviera model and strongly restricted permutations of length $n+4$ where $W = \{-2, -1, 2\}$.

configurations $c_1 \ldots c_n$ of length n and the configurations of length $n+12$ starting with 100110 and ending with 011001 which, although not maximal (because of the boundary condition), do not properly contain[3] any substrings forbidden by Lemma 1.1. Each of those extended configurations can, therefore, be represented as a walk on $\mathcal{G}_{\mathcal{R}}$ (Figure 1) starting at the node 100 and ending at the node 001. Conversely, if a configuration $100110c_1 \ldots c_n011001$ (of length $n+12$) does not properly contain any substrings forbidden by Lemma 1.1, or equivalently, can be represented as a walk on $\mathcal{G}_{\mathcal{R}}$ (of length $n+9$) starting at 100 and ending at 001, then after removing prefix 100110 and suffix 011001 one is left with a proper maximal configuration $c_1 \ldots c_n$ of length n. This is because removing prefix and suffix corresponds to cutting off the first part of the walk 100-001-011-110-$10c_1$-$0c_1c_2$ and the last part of the walk $c_{n-1}c_n0$-c_n01-011-110-100-001. Note that regardless of what c_1, c_2, c_{n-1}, and c_n are — the next node after $0c_1c_2$ as well as the node just before $c_{n-1}c_n0$ will always have to be one of the starting/ending nodes, which means that the remaining part of the walk encodes a proper maximal configuration $c_1c_2 \ldots c_{n-1}c_n$.

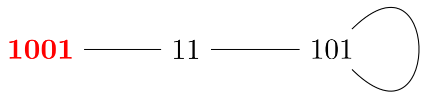
3 The Riviera model and closed walks on P_3 with a loop

In (1.4) we have derived the generating function $h(x)$ for the number of maximal Riviera configurations (of variable length) containing a fixed number k of occupied lots. This sequence appears in OEIS [9] in two instances as A052547 with offset 3 and as A096976 with offset 5. There are three more related sequences: A006053, A028495, and A096975, satisfying the same recurrence relation with different initial conditions. Each of these sequences is connected to the number of walks on the graph P_3 (the path graph over

[3]By properly contain we mean contained as a substring within an unpadded configuration.



(a) P_3 with a loop, $\mathcal{P}$ for short



(b) $\mathcal{P}$ with labeled nodes

Figure 4

three nodes) with a loop added at one of the end nodes. This graph is represented in Figure 4(a) and we denote it by $\mathcal{P}$. The precise connection relating this graph with our sequence is given in the following theorem.

Theorem 3.1. *The number of maximal Riviera configurations containing exactly k occupied lots is equal to the number of closed walks of length $k+4$ on the graph $\mathcal{P}$ which start and end at the node of degree 1. There is a natural bijection relating these quantities.*

Remark 3.2. If equipotency result is all one wishes to pursue, it suffices to compare the two sequences' generating functions. Note that

$$\begin{aligned}\frac{1-x-x^2}{1-x-2x^2+x^3} &= 1+x^2+x^4+x^4\cdot\frac{1+2x^2-x^3}{1-x-2x^2+x^3}\\ &= 1+x^2+x^4+x^4\cdot h(x),\end{aligned}$$

where $\dfrac{1-x-x^2}{1-x-2x^2+x^3}$ is the generating function of the sequence A096976 and $h(x)$ is the generating function for the number of maximal Riviera configurations (of variable length) containing a fixed number k of occupied lots, see (1.4).

Proof of Theorem 3.1. From Lemma 1.1 we know that no three consecutive 0's are allowed in a maximal configuration. That means that each two neighboring 1's must be separated by zero, one or two 0's. This further means that, after ignoring leading and trailing 0's, each maximal configuration can be identified by a sequence of strings in the set $\{11, 101, 1001\}$. We assume here that the last 1 in one string overlaps with the first 1 in the next. E.g. we would split the configuration 11011001101 as 11-101-11-1001-11-101.
From Lemma 1.1 we also see that 11 cannot be followed or preceded by 11 (as this would produce 111); 101 and 1001 cannot be followed or preceded by 1001 (as this would produce 0100 or 0010). It is easy to see that the remaining transitions: 1001-11 and 11-101 going in either direction, and the

loop at 101 — can all appear in a maximal configuration and are, thus, all allowed. These transitions are shown in the node-labeled graph $\mathcal{P}$ in Figure 4(b). We use undirected edges as in each case the transitions going either way are allowed.

Consider now the mapping which to each maximal Riviera configuration $c_1 \dots c_n$ assigns the configuration $100110c_1 \dots c_n011001$. By Remark 2.2 we know that this map is a bijection from the set of all maximal Riviera configurations with exactly k occupied lots to the set of configurations of the form $100110c_1 \dots c_n011001$ which have exactly $k+6$ occupied lots. Those obtained configurations are not maximal but do not properly contain substrings forbidden by Lemma 1.1.

Now each of those configurations of the form $100110c_1 \dots c_n011001$, where $c_1 \dots c_n$ is a proper maximal configuration with k occupied lots, can be represented as a walk of length $k+4$ on the graph $\mathcal{P}$ in Figure 4(b) which starts and ends at 1001. Conversely, one easily checks (by inspecting all length 2 walks) that a walk on this graph can never produce a configuration properly containing a substring which is forbidden by Lemma 1.1. Therefore, each walk of length $k+4$ starting and ending at 1001 will necessarily produce a configuration of the form $100110c_1 \dots c_n011001$ which does not properly contain a substring forbidden by Lemma 1.1 and has $k+6$ 1's. By Remark 2.2, the word $c_1 \dots c_n$ will be a proper maximal configuration with exactly k 1's.

Putting everything together gives us the required bijection. A maximal Riviera configuration $c_1 \dots c_n$ containing exactly k occupied lots is written as the string $100110c_1 \dots c_n011001$, which is then represented as a walk of length $k+4$ over the graph $\mathcal{P}$. As an example, the maximal configuration 10110 is mapped to $10110 \to 100110|10110|011001$ which corresponds to the walk: $1001 \to 11 \to 101 \to 101 \to 11 \to 1001 \to 11 \to 1001$ which begins and ends with the node 1001. $\square$

4 Concluding remarks

Modern enumerative combinatorics uses a wide and evergrowing repertoire of methods and techniques. Prominent among them are analytical methods, based on extracting the information encoded in generating functions of the considered sequences. Another example are the methods based on transfer matrices. Powerful and useful as they are, in the eye of many practitioners they both suffer from a serious shortcoming: They do not reveal the true nature of connections between different combinatorially interesting families. According to their opinion, only a combinatorial, bijective, proof can offer deep(er) insight. So, combinatorial proofs are often sought even long after

interesting results have been obtained in other ways.
In this short note we provide combinatorial proofs for two enumerative results from our earlier paper on settlement planning [3]. We have constructed bijections connecting jammed configurations in a 1-dimensional toy model of settlement planning with, on one hand, restricted permutations, and, on the other hand, walks on a small graph. Hence, they reveal hidden connections between seemingly unrelated families of objects, providing support for the above claim and justifying the effort invested in their construction.
Our toy model, simple as it is, still offers many interesting problems which are still unanswered. For example, it would be interesting to (numerically) simulate its progression inwards, once the sea-front is fully developed (i.e., when no new constructions are legally possible). Another direction, more suitable for analytical and/or combinatorial treatment, would be to remain in one dimension but to allow multi-storied buildings and to investigate whether this modification results in more efficient jammed configurations. We are confident that the interested reader will find also problems and research questions we have overlooked.

Acknowledgments

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Plotting planar and toroidal maps

Mathieu Dutour Sikirić

Abstract

Plotting planar and toroidal maps is a common problem in graph visualization. We present here an algorithmic implementation that has been used in [3]. The implementation is publicly available.

1 Introduction

The graphs considered in this study can have multiple edges, and it is also possible that both endpoints of a face are the same. A graph is said to be of genus g if it can be embedded on a surface of genus g. If $g = 0$, the graph is called *planar* and if $g = 1$, it is called *toroidal*. A *face* in such an embedding is bounded by a sequence of edges and is homeomorphic to a disk. This embedding gives positions of vertices, edges, and faces.

The combinatorial structure of the vertices, edges, and faces can be studied without having a specific embedding, as the individual positions of the vertices do not matter very much. However, if one has built a graph with edges, vertices, and faces, we may want to find an embedding, as this can be helpful for scientific purposes and also for visualization.

The problem of graph embedding has been considered from various viewpoints. In [9] a method to use the eigenvectors of the largest eigenvalues of the adjacency matrix was used. The method works well for plane graphs but suffers from one key problem: the inner faces tend to be very small and not visible. Also, the method works only for planar graphs.

The program CaGe (see [1, 2]) uses an iterative process in order to get an embedding. A priori, it works only for planar graphs. Another class of

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methods is to minimize the functional F

$$F = \sum_{e=(i,j)\in E(G)} f(\|x_i - x_j\|)$$

A more powerful and conceptual technique is to use primal-dual circle packing for obtaining a drawing (see [6]). The technique works for any genus and graphs that are 3-connected.

2 Combinatorial representation of maps

A general 2-dimensional map $\mathcal{M}$ can be represented by vertices, edges, and faces. The incidence relation between those objects is best encoded by the notion of flags. A flag is a triple (v, e, F) with v a vertex, e an edge, F a face and $v \in e \subset F$. Denote by $Flag(\mathcal{M})$ the set of all flags of $\mathcal{M}$. From the flags, we can define the flag operators σ_0, σ_1, and σ_2. For a flag $f = (v, e, F)$, the image $\sigma_0(f)$ is defined as the flag that is identical to f except at the vertex, $\sigma_1(f)$ the flag that is identical to f except at the edge and $\sigma_2(f)$ the flag that is identical to f except at the face. The flag operators are uniquely determined by those constraints. They satisfy the relation $\sigma_i^2 = Id$ and $\sigma_0\sigma_2 = \sigma_2\sigma_0$.

The flag operators are permutations on the set of flags, and they can be used for representing the maps efficiently. For example, the vertices correspond to the orbits of the group generated by $\{\sigma_1, \sigma_2\}$ on $Flag(\mathcal{M})$. The formalism of flags works well even for the non-orientable maps such as the projective plane or the Klein bottle.

However, for the purpose of this work, the maps that we consider are oriented. So instead of something so general, we use the directed edge formalism. At every vertex and every edge, we associate a direction. We define two operators:

1. The next operator n that sends a directed edge to the next one in the positive (counterclockwise) orientation.

2. The inverse operator i that reverses the direction of the directed edge.

See Figure 1 for an example of such an action on a directed edge. Since a planar graph has a finite number of directed edges, those operators can be represented as two permutations. The vertices, edges, and faces correspond to the orbits of the group generated by $\{n\}$, $\{i\}$, and $\{n \circ i\}$.

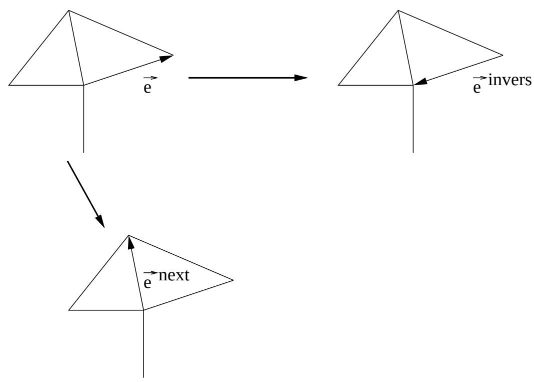


Figure 1: The action of the next and inverse operator on a directed edge

3 The primal-dual circle method

It is a remarkable fact that in dimension 2, we have a meta-theorem that combinatorics=geometry. That is, combinatorics alone suffices to encode an object. One example of such a theorem is Steinitz theorem ([8] and [10, Chapter 4]) that states that a graph is the skeleton of a 3-dimensional polytope if and only if it is planar and 3-connected. The primal-dual circle packings are an effective technique that allows to represent graphs given by their combinatorial information (see Chapter 5 of [5]).

The idea is to put circles $C_{vert}(v)$ in the vertex center and face center $C_{face}(F)$ so that:

- The interiors of the circles $C_{vert}(v)$ are all distinct. Two circles $C_{vert}(v)$ and $C_{vert}(v')$ intersect in a point if and only if the vertices v and v' are adjacent. Denote $P_{vert}(v, v')$ the intersection point.
- The interiors of the circles $C_{face}(F)$ are all distinct. Two circles $C_{face}(F)$ and $C_{face}(F')$ intersect in a point if and only if the faces F and F' share an edge. Denote $P_{face}(F, F')$ the intersection point.
- If v and v' are adjacent vertices with an edge e shared between faces F and F' then we have $P_{vert}(v, v') = F_{face}(F, F')$

Those properties are represented in Figure 2.

According to [6], if a graph G of genus g is 3-connected and simply connected then there exists an embedding of the universal cover of G into the sphere (if $g = 0$), the plane (if $g = 1$) or the hyperbolic plane (if $g > 1$). See Figure 3 for one such example.

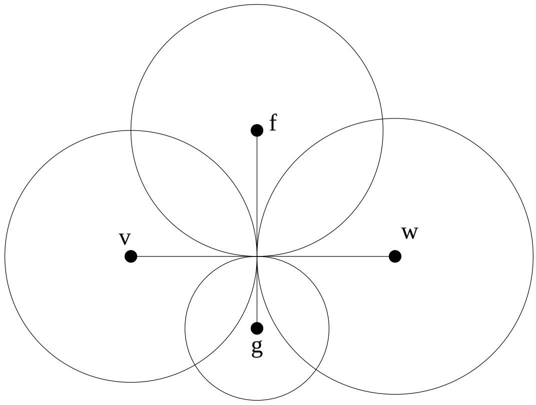


Figure 2: The local picture of a primal-dual circle representation

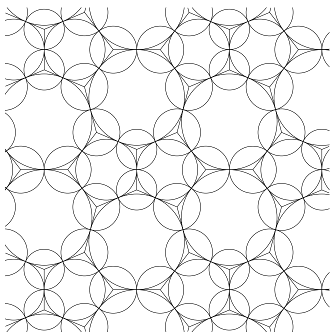


Figure 3: The edges, circle, and face circles of a primal-dual representation

Due to the invariance by duality of the primal-dual circle packing, it makes sense to consider the medial graph $Med(G)$ for a graph of genus g. It is the graph whose vertices are the vertices and faces of G, and two vertices of $Med(G)$ are adjacent if one is a vertex u of G, and another is a face F of G and $u \in F$. The graph $Med(G)$ is also of genus g and its faces are of size 4. This is the method used in this work.

4 Numerical techniques for the planar case

The equation system that needs to be resolved for plotting the graph represents the fact that the sum of the angles is 2π at each vertex. The angles are computed from the radii of the circles $C_{vert}(u)$ and $C_{face}(F)$

In the case of genus $g = 1$, the angle equations are simpler to write down since we do not have to deal with spherical or hyperbolic trigonometry. We define for each vertex v of $Med(G)$ the angle sum:

$$\phi_v = \sum_{uv \in E(Med(G))} \arctan\left(\frac{r_u}{r_v}\right).$$

The equations to resolve are thus

$$\phi_v = \pi \text{ for } v \in V(G).$$

This is a system of non-linear equations, and we have the equation

$$\sum_{v \in V(Med(G))} \phi_v - \pi = 0. \tag{4.1}$$

This conservation equation renders the system underdetermined. Thus, if the collection (r_u) is a solution, then (λr_u) is also a solution.

[6] has given an algorithm for computing the primal dual circle packings radii. It consists of computing the defect at every node and increasing/decreasing the radius value according to $\phi_v > \pi$ or $\phi_v < \pi$. It is a geometric method that is quite efficient, but in some cases it is very slow.

Newton's method cannot be applied right away as the problem is underdetermined. We address this by restricting ourselves to the vector space of directions that do not change the sum of the radius of all the circles. Another more essential problem is that when we apply the Newton method, we can obtain a negative value for some radii. The technique that we use is instead to rescale the increment by a factor c in the following way:

$$x^{(n+1)} = x^{(n)} - c\frac{f(x^{(n)})}{f'(x^{(n)})} \text{ with } 0 < c \leq 1$$

In the terminology of [7] it is Newton method with a line search. We start with a factor $c = 1$ and then decrease it by a factor 1.2 until:

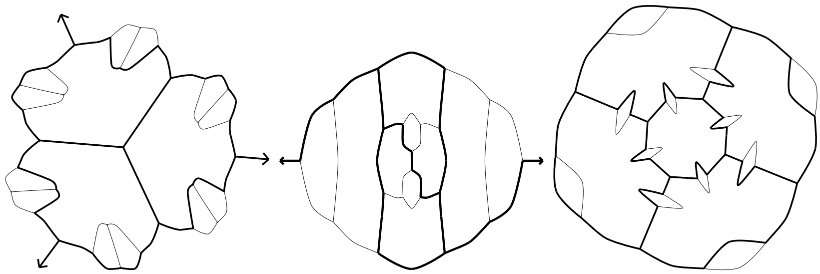


Figure 4: The k-fold axis position can be on a vertex, an edge, or a face

1. All the radii are positive.
2. The sum of the absolute error $\sum_v (\phi_v - \pi)^2$ has decreased.

This turns out to work very well.

5 Graphical niceties

Obtaining a graphical representation of a map can be helpful. However, we sometimes want something nicer to look at. For example, if a graph has a 3-fold axis of symmetry, then we may want to have this axis directly visible on the map. If the axis is passing through a face, then we just need to select the appropriate face. If, however, the axis passes through two vertices, then we have a problem. We need to represent it, but we must choose the correct axis. It is also possible for a 2-fold axis to pass through two edges. Some examples are shown in Figure 4.

The primal-dual technique requires 3-connectivity and will not work with 2-gons and 1-gons. The technique is to refine it. First, for a map M we replace it with the order complex map $Ord(M) = Trunc(Med(M))$. Then we insert a vertex on each edge. Finally, we put a vertex on each face and connect it to all incident vertices. The resulting triangulation is 3-connected.

6 CaGe process and the integrated algorithm

The algorithm of [1] is used for the drawing of planar graphs. It works in the following way:

- Select an external face F and puts the vertices in a circle.
- Put a point x_F in the center of each face that is not external. We have a collection of points $(x_v) \cup (x_F)$.

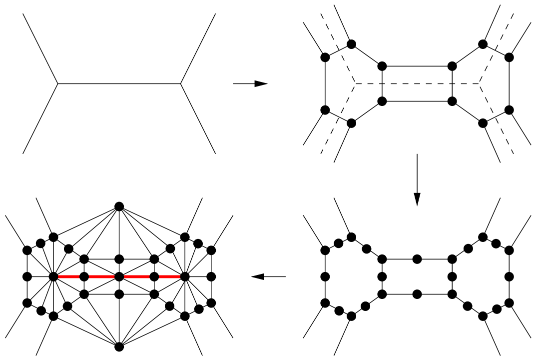


Figure 5: The scheme used for transform a graph into another that is easier to draw.

- For each point x that is not on the outer circle and that is contained in the triangles (x, x^i, x^{i+1}), we have

$$x = \frac{1}{\sum_{i=1}^{m} A_T^2(x, x^i, x^{i+1})} \sum_{i=1}^{m} A_T^2(x, x^i, x^{i+1}) \frac{x + x^i + x^{i+1}}{3}$$

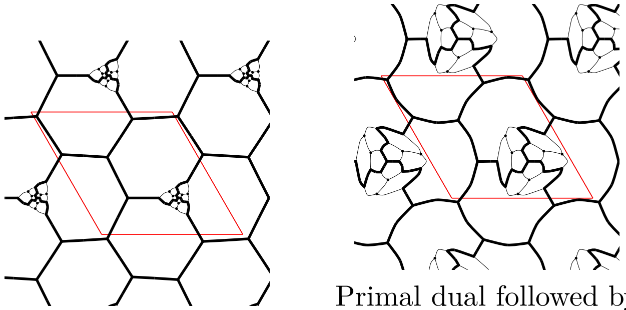
  with $A_T(x, y, z)$ the area of the triangle of vertices x, y and z.

- The equation can be solved by fixed-point iterations.

For a planar graph, we apply the following construction:

- Apply the truncation scheme of Section 5.
- Depending on the choice of the external face (vertex, edge, or face), select the corresponding external face.
- Apply the CaGe process to find an embedding.
- From the positions of the various points, build the path that corresponds to an edge of the graph.

The CaGe process is an averaging process where we average the position of the neighbor in order to update the position of a point. This same process can be applied in the toroidal case. This can yield a clear improvement to the obtained graph as shown in Figure 6.



Primal dual

Primal dual followed by CaGe process

Figure 6: The graph of a toroidal map obtained by the primal-dual processed followed by the one with the CaGe process applied afterward

7 The effective design of the software and how to use it

It is important to have Open Source code so that other users can benefit from the created software. But, that is not quite sufficient if the Open source software is very difficult to use. Thus, we have tried to make a Python-based solution which should be convenient as Python has emerged as the equivalent of English in Computer Science: not a perfect language, but everybody is using it.

The computation of the coordinates is done with a code written in **C++**. The matrix algebra is done by using the Eigen library ([4]). The input file of the program follows the Fortran Namelist format, which is a simple yet reasonable to use format.

The data are then exported to a `SVG` file. The `SVG` file format is adequate for this purpose since it is a symbolic text file format that can be used in Web browsers. Also, `SVG` files can be edited by **inkscape**. This allows users to easily edit according to their wishes.

The Python code is accessed via

```
pip3 install https://github.com/MathieuDutSik/PyPlot_orientedmap
```

Alas, not everything is so simple with the program usage. The difficulty is in creating the input. As convenient as the format with directed edges is, it has a distinct computational aspect to it.

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A New Theorem from the Number Theory and its Application for a 3-adic Valuation for Large Schröder Numbers

Jovan Mikić

Abstract

By using central Delannoy numbers and a new theorem from number theory, we give almost a complete 3-adic valuation of large Schröder numbers S_n, except for the case S_{6k+4}, $k \geq 0$.

Keywords: Large Schröder number, Central Delannoy number, 3-adic valuation, Catalan number, Kummer's theorem.
2020 ***Mathematics Subject Classification***: 05A10, 11B65.

1 Introduction

The large Schröder Numbers S_n count all lattice paths in the plane from $(0,0)$ to (n,n) by using horizontal steps $(1,0)$, vertical steps $(0,1)$, and diagonal steps $(1,1)$ such that never rise above the main diagonal $y = x$. It is known [6, Exercise 6.39], at least, 11 combinatorial objects are counted by large Schröder numbers S_n.
A formula for calculating large Schröder numbers S_n is also well-known:

$$S_n = \sum_{k=0}^{n} \frac{1}{k+1} \binom{n+k}{k} \binom{n}{k}. \tag{1.1}$$

This sequence starts with: $S_0 = 1$, $S_1 = 2$, $S_2 = 6$, $S_3 = 22$, $S_4 = 90$, $S_5 = 394$, $S_6 = 1806, \ldots$; and it can be found as sequence $A006318$ in [5]
Recently, a new formula for a 3-adic valuation of large Schröder numbers S_n has been discovered, where the p-adic valuation of an integer $y \geq 0$ is

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definied as follows: Let p be a prime number, and let y be an integer. Then $v_p(y)$ denotes the exponent of the highest power of the prime number p that divides y, i.e., $p^{v_p(y)}$ divides y, $p^{v_p(y)+1}$ does not divide y, and is called the p-adic valuation of y.
It is readily verified that $v_3(S_0) = v_3(S_1) = v_3(S_3) = v_3(S_5) = 0$, $v_3(S_2) = v_3(S_6) = 1$, and $v_3(S_4) = 2$.
Furthermore, it is well-known that Catalan number C_n counts all lattice paths in the plane from $(0,0)$ to (n,n) by using horizontal steps $(1,0)$ and vertical steps $(0,1)$ that never rise above the main diagonal $y = x$. Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ represent the famous sequence which has the most applications in combinatorics after the binomial coefficients.
This sequence starts with: $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$, $C_5 = 42$, $C_6 = 132$, $\ldots$; and it can be found as sequence $A000108$ in [5]. It is readily verified that $v_3(C_0) = v_3(C_1) = v_3(C_2) = v_3(C_3) = v_3(C_4) = 0$, and $v_3(C_5) = v_3(C_6) = 1$.
Now, we are ready to present a recently [4, Theorem 6, Eqns. (28) and (29), p. 6] discovered formula for a 3-adic valuation of large Schröder numbers S_n:

Theorem 1.1. *Let n be a non-negative integer. Then the following formulae*

$$v_3(S_{2n+1}) = v_3(C_n), \tag{1.2}$$
$$v_3(S_{2n+2}) = 1 + v_3(2n+1) + v_3(C_n). \tag{1.3}$$

hold.

The Equation (1.2) represents, by our opinion, one of the most beautiful results in number theory. It tells us that large Schröder numbers with odd indices have the same factorization of powers of three as Catalan numbers C_n.
Recently, the proof of Theorem 1.1 was given using the little Schröder numbers s_n. The little Schröder numbers s_n count, among other things, the number of plane trees with a given set of leaves, the number of ways of inserting parentheses into a sequence, and the number of ways of dissecting a convex polygon into smaller polygons by inserting diagonals. Furthermore, if n is a natural integer, then it is known that $s_n = \frac{1}{2} \cdot S_n$.
This sequence starts with: $s_0 = 1$, $s_1 = 1$, $s_2 = 3$, $s_3 = 11$, $s_4 = 45$, $s_5 = 197$, $s_6 = 903$, $\ldots$; and it can be found as sequence $A001003$ in [5]. Obviously, $v_3(s_n) = v_3(S_n)$, for any non-negative integer n.
The aim of this paper is to give an another proof of Theorem 1.1. We give almost a complete proof of Theorem 1.1 by using central Delannoy numbers D_n and a new theorem from number theory.

The central Delannoy numbers D_n represent the number of lattice paths in the plane from $(0,0)$ to (n,n) by using horizontal steps $(1,0)$, vertical steps $(0,1)$, and diagonal steps $(1,1)$. Such paths are also known in the literature as royal paths. Sulanke [7] gave, in a recreational spirit, a collection of 29 configurations counted by these numbers.

For calculating central Delannoy numbers D_n, there are, at least, two formulae [1, Eq. (1), p. 1]:

$$D_n = \sum_{k=0}^{n} \binom{n}{k}^2 2^k, \tag{1.4}$$

$$D_n = \sum_{k=0}^{n} \binom{n+k}{k}\binom{n}{k}. \tag{1.5}$$

This sequence starts with: $D_0 = 1$, $D_1 = 3$, $D_2 = 13$, $D_3 = 63$, $D_4 = 321$, $D_5 = 1683$, $D_6 = 8989$, $\ldots$; and it can be found as sequence $A001850$ in [5]. It is readily verified that $v_3(D_0) = v_3(D_2) = v_3(D_6) = 0$, $v_3(D_1) = v_3(D_4) = 1$, and $v_3(D_3) = v_3(D_5) = 2$.

It is known [4, Eq. 5, p. 2] that central Delannoy numbers satisfy the following second order recurrence relation:

$$(n+2)D_{n+2} = 3(2n+3)\cdot D_{n+1} - (n+1)\cdot D_n. \tag{1.6}$$

A connection between central Delannoy numbers and large Schröder numbers is given [4, Eq. 6, p. 2] by the following formula:

$$S_n = \frac{1}{2}\cdot(-D_{n-1} - D_{n+1} + 6D_n), \tag{1.7}$$

where n is a natural number.

Finally, we present a recently discovered new theorem from number theory. Let x and y be integers such that $x \neq 0$ and $x \neq \pm 1$. Let $\omega_x(y)$ denote the exponent of the highest power of an integer x that divides an integer y. Note that we shall use the notation $v_p(y)$ instead of $\omega_x(y)$ when the integer x is equal to a prime number p.

Let n be a non-negative integer. Let a and b be integers such that are relatively prime and $a \neq -b$, as well as $a + b \neq \pm 1$.

The recently discovered theorem [4, Theorem 1, Eqns. (18) and (19)] from number theory states that

Theorem 1.2.

$$\omega_{a+b}(\sum_{k=0}^{2n} \binom{2n}{k}^2 \cdot a^{2n-k} \cdot b^k) = \omega_{a+b}(\binom{2n}{n}), \tag{1.8}$$

$$\omega_{a+b}(\sum_{k=0}^{2n+1} \binom{2n+1}{k}^2 \cdot a^{2n+1-k} \cdot b^k) = 1 + \omega_{a+b}((2n+1) \cdot \binom{2n}{n}). \tag{1.9}$$

By setting $a := 2$ and $b := 1$ in (1.8) and (1.9), we obtain the formulae for calculating a 3-adic valuation of central Delannoy numbers [4, Theorem 5, Eqns. (27) and (26), p.6], where $v_3(y) = \omega_3(y)$, as we said before (see also [2]). Therefore, the following two formulae are true for any non-negative integer n:

$$v_3(D_{2n}) = v_3(\binom{2n}{n}), \tag{1.10}$$

$$v_3(D_{2n+1}) = 1 + v_3(2n+1) + v_3(\binom{2n}{n}). \tag{1.11}$$

Recently, Lengyel gave, among other, the formula [1, Theorem 10, p. 8] for 3-adic valuation of central Delannoy numbers. However, his formula is true for any n sufficiently large. Our two formulae from Eqns. (9) and (10) are true for any natural number n.
Lengyel also gave the formula [1, Theorem 17, p. 19] for the 3-adic valuation of large Schröder numbers S_n for the case $n-1$ is divisible by 3.
The rest of the paper is structured, as follows: in the second section, we present auxiliary results. In the third section, we give a proof of (1.2). In the fourth section, we give a proof of (1.3) for the cases $n-2$ is divisible by 3 and n is divisible by 3.

2 Auxiliary Results

Our first auxiliary result is:

Proposition 2.1. *Let n be a natural number. Then the following recurrence relation holds:*

$$2(n+1)S_n = 3D_n - D_{n-1}. \tag{2.1}$$

The second auxiliary result is:

Proposition 2.2. *Let n be a non-negative integer. Then the following recurrence relation holds:*

$$2(2n+1)S_n = D_{n+1} - D_{n-1}. \tag{2.2}$$

Finally, the third auxiliary result is:

Proposition 2.3. *Let k be a non-negative integer. Then the two following equations hold:*

$$v_3(\binom{6k+2}{3k+1}) = v_3(\binom{6k}{3k}) = v_3(\binom{2k}{k}) \tag{2.3}$$

$$v_3(\binom{6k+4}{3k+2}) = 1 + v_3(2k+1) + v_3(\binom{2k}{k}) \tag{2.4}$$

Proofs of (2.1) and (2.2) easily follow from (1.6) and (1.7). Furthermore, the proof of (2.3) follow from Kummer's theorem [3]. Namely, we recall that Kummer's theorem states that for given integers $n \geq m \geq 0$ and a prime number p, the p-adic valuation of the binomial coefficient $\binom{n}{m}$ is equal to the number of carryovers that occur when m and $n-m$ are added in base p.

Finally, the proof of (2.4) follows from (2.3) and by setting $n = 3k+1$ in the well-known equation:

$$\binom{2n+2}{n+1} = 2(2n+1)C_n; \tag{2.5}$$

where n is a non-negative integer.

Obviously, by the definition of Catalan numbers, it follows that

$$v_3(C_n) = v_3(\binom{2n}{n}) - v_3(n+1).$$

Therefore, we leave the proofs of the auxiliary results to the readers.

3 A Proof of the Eq. (1.2)

Substituting $2n+1$ for n in (2.1), we get

$$4(n+1)S_{2n+1} = 3D_{2n+1} - D_{2n}, \tag{3.1}$$

where n is a non-negative integer.

From (3.1) it follows that

$$v_3(n+1) + v_3(S_{2n+1}) = v_3(3D_{2n+1} - D_{2n}). \tag{3.2}$$

By (1.10) and (1.11), we know that

$$v_3(3 \cdot D_{2n+1}) = 1 + v_3(D_{2n+1}) > 1 + v_3(D_{2n}) > v_3(D_{2n}). \tag{3.3}$$

From (3.3) it follows that

$$v_3(3D_{2n+1} - D_{2n}) = v_3(D_{2n}). \tag{3.4}$$

Furthermore, by (3.4), (3.2) becomes

$$v_3(n+1) + v_3(S_{2n+1}) = v_3(D_{2n}). \tag{3.5}$$

Finally, by (1.10), we have gradually:

$$v_3(n+1) + v_3(S_{2n+1}) = v_3(\binom{2n}{n}), \tag{3.6}$$

$$v_3(S_{2n+1}) = v_3(\binom{2n}{n}) - v_3(n+1), \tag{3.7}$$

$$v_3(S_{2n+1}) = v_3(\frac{1}{n+1} \cdot \binom{2n}{n}), \tag{3.8}$$

$$v_3(S_{2n+1}) = v_3(C_n). \tag{3.9}$$

The last equation proves the assertion (1.2).

4 A Proof of (1.3)

We give a proof of (1.3) for the cases $n-2$ is divisible by 3 and n is divisible by 3.
Substituting $2n+2$ for n in (2.2), we get

$$2(4n+5)S_{2n+2} = D_{2n+3} - D_{2n+1}, \tag{4.1}$$

where n is a non-negative integer.
By (4.1), it follows that

$$v_3(4n+5) + v_3(S_{2n+2}) = v_3(D_{2n+3} - D_{2n+1}). \tag{4.2}$$

The first case:
Let $n-2$ be divisible by 3. Then $n = 3k+2$, where k is a non-negative integer. Since $v_3((4(3k+2)+5) = v_3(12k+13) = 0$, we get from equation (4.2):

$$v_3(S_{6k+6}) = v_3(D_{6k+7} - D_{6k+5}). \tag{4.3}$$

By (1.11) and Kummer's theorem, it can be shown that

$$v_3(D_{6k+7}) = 1 + v_3(2k+1) + v_3(\binom{2k}{k}) - v_3(k+1). \tag{4.4}$$

Furthermore, by (1.11), it follows that

$$v_3(D_{6k+5}) = 2 + v_3(2k+1) + v_3(\binom{2k}{k}). \tag{4.5}$$

By Eqns. (4.4) and (4.5), it follows that $v_3(D_{6k+7}) < v_3(D_{6k+5})$. Therefore, we conclude that

$$v_3(D_{6k+7} - D_{6k+5}) = v_3(D_{6k+7}). \tag{4.6}$$

By using (4.4) and (4.6), (4.3) becomes

$$v_3(S_{6k+6}) = 1 + v_3(2k+1) + v_3(\binom{2k}{k}) - v_3(k+1). \tag{4.7}$$

By setting $n = 3k+2$ in (1.3), the equation (1.3) gradually becomes

$$\begin{aligned} v_3(S_{6k+6}) &= 1 + v_3(6k+5) + v_3(C_{3k+2}), \\ &= 1 + v_3(\binom{6k+4}{3k+2}) - v_3(3k+3), \\ &= 1 + (1 + v_3(2k+1) + v_3(\binom{2k}{k})) - 1 - v_3(k+1), \\ &= 1 + v_3(2k+1) + v_3(\binom{2k}{k}) - v_3(k+1). \end{aligned} \tag{4.8}$$

By using (4.7) and (4.8), it follows that (1.3) is true if $n = 3k+2$. This completes the proof of the first case.

The second case:

Let n be a non-negative integer divisible by 3. Then $n = 3k$, where $k \geq 0$. By setting $n = 3k$ in (4.1), we obtain that

$$v_3(S_{6k+2}) = v_3(D_{6k+3} - D_{6k+1}). \tag{4.9}$$

By using (1.11) and (2.4), it can be shown that

$$\begin{aligned} v_3(D_{6k+3}) &= 1 + v_3(6k+3) + v_3(\binom{6k+2}{3k+1}), \\ &= 2 + v_3(2k+1) + v_3(\binom{2k}{k}). \end{aligned} \tag{4.10}$$

On the other hand, by using (1.11) and (2.3), it can be shown that

$$v_3(D_{6k+1}) = 1 + v_3(\binom{2k}{k}). \tag{4.11}$$

By using (4.10) and (4.11), it follows that $v_3(D_{6k+1}) < v_3(D_{6k+3})$. Hence, we conclude that

$$v_3(D_{6k+3} - D_{6k+1}) = v_3(D_{6k+1}). \tag{4.12}$$

Therefore, by using (4.12), it follows that

$$v_3(S_{6k+2}) = 1 + v_3(\binom{2k}{k}). \tag{4.13}$$

By setting $n = 3k$ in (1.3), the equation (1.3) gradually becomes

$$\begin{aligned} v_3(S_{6k+2}) &= 1 + v_3(6k+1) + v_3(C_{3k}), \\ &= 1 + v_3(\binom{6k}{3k}) - v_3(3k+1), \\ &= 1 + v_3(\binom{2k}{k}). \end{aligned} \tag{4.14}$$

By using (4.13) and (4.14), it follows that (1.3) is true for $n = 3k$. This completes the proof of the second case.

Remark 4.1. The case $n-1$ is divisible by 3 of (1.3) must be treated with another approach (see, for example, [1, Theorem 17, p. 19] or [4, Section 12, p. 25]) due to the fact that

$$v_3(D_{6k+5}) = v_3(D_{6k+3}) = 1 + v_3(D_{6k+4}). \tag{4.15}$$

We conjecture that the following equation is true:

$$v_3(3D_{6k+4} - D_{6k+3}) = v_3(D_{6k+3}), \tag{4.16}$$

where $k \geq 0$.

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On two sequences and their hypersequences

Daniele Parisse

Abstract

We study the hypersequences $\left(a_n^{(r)}\right)_{n\in\mathbb{N}_0}$ and $\left(b_n^{(r)}\right)_{n\in\mathbb{N}_0}$, $r \in \mathbb{N}_0$, of the two sequences $a_n := (-1)^n$ and $b_n := (-1)^{n+1}n, n \in \mathbb{N}_0$. First, we show the relationship between these hypersequences. Subsequently, we prove that both the rth rows and the nth columns of the arrays $\left(a_n^{(r)}\right)$ and $\left(b_n^{(r)}\right)$, $r, n \in \mathbb{N}_0$, satisfy linear recurrence relations. This yields alternative representations of $a_n^{(r)}$ and $b_n^{(r)}$. Finally, we determine their ordinary generating functions and the recurrence relations of two special subsequences.

Keywords: Hypersequences; recurrences; binomial coefficients; Stirling numbers of the first kind; ordinary generating functions; Catalan numbers.
2020 ***Mathematics Subject Classification***: 05A10, 05A15, 11B37.

1 Introduction

The sequence $(c_n) = (0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots)$ is the sequence A001057 in the On-Line Encyclopedia of Integer Sequences (OEIS ®) [3]. It is the sequence of all integers and can be described as follows: start from 0 and go forward and backward with increasing step sizes. Accordingly, the sequence can be defined by

$$c_{2n} = -n, \quad c_{2n+1} = n + 1, \quad n \geq 0, \tag{1.1}$$

showing that the function $c \colon \mathbb{N}_0 \to \mathbb{Z}$, $n \mapsto (-1)^{n+1} \cdot \left\lfloor \frac{n+1}{2} \right\rfloor$ is a bijection. Since the successor function $s \colon \mathbb{N}_0 \to \mathbb{N}$, $n \mapsto n + 1$, is also a bijection, this shows that $\mathbb{N}$, $\mathbb{N}_0$, and $\mathbb{Z}$ have the same cardinality, namely $\aleph_0$, the first transfinite cardinal number.

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Lemma 1.1. *The sequence* $(c_n)_{n\in\mathbb{N}_0}$ *satisfies the recurrences*

$$c_0 = 0, \quad c_{n+1} = c_n + (n+1)(-1)^n, \quad n \geq 0, \tag{1.2}$$

and

$$c_0 = 0, \quad c_{n+1} = -c_n + \frac{1}{2}(1 + (-1)^n), \quad n \geq 0, \tag{1.3}$$

which have the solution

$$c_n = \sum_{k=0}^{n}(-1)^{k+1}k = \frac{1}{4}(1-(2n+1)(-1)^n) = 2\cdot\left(\left\lfloor\frac{n+1}{2}\right\rfloor\right)^2 - \binom{n+1}{2}, \tag{1.4}$$

for $n \geq 0$.

Proof. By definition, we have $c_0 = 0$. Let n be even, that is $n = 2m$, $m \in \mathbb{N}_0$. Then, by (1.1) we have $c_{n+1} = c_{2m+1} = m + 1$, and $c_n = c_{2m} = -m$. Therefore, $c_n + (n+1)(-1)^n = c_{2m} + (2m+1)(-1)^{2m} = -m + (2m+1) = m+1 = c_{2m+1} = c_{n+1}$. Now, let n be odd, that is $n = 2m+1$, $m \in \mathbb{N}_0$. Then, again by (1.1) we have $c_{n+1} = c_{2m+2} = -(m+1)$, and $c_n = c_{2m+1} = m+1$. Hence, $c_{n+1} - c_n = c_{2m+2} - c_{2m+1} = -(m+1) - (m+1) = -(2m+2) = (2m+2)\cdot(-1)^{2m+1} = (n+1)\cdot(-1)^n$, and these two cases prove (1.2).
Let n be even, that is $n = 2m$, $m \in \mathbb{N}_0$. Then, by (1.1) we have $c_{n+1} + c_n = c_{2m+1} + c_{2m} = m+1+(-m) = 1 = (1+(-1)^{2m})/2$. Similarly, for n odd, that is $n = 2m+1$, $m \in \mathbb{N}_0$, we have again by (1.1) $c_{n+1} + c_n = c_{2m+2} + c_{2m+1} = -(m+1) + m + 1 = 0 = (1 + (-1)^{2m+1})/2$, and this proves (1.3).
The solution of the recurrence (1.2) can be obtained by the method of backward substitution and noting that $c_0 = 0$

$$\begin{aligned}
c_n &= c_{n-1} - n\cdot(-1)^n \\
&= c_{n-2} - (n-1)\cdot(-1)^{n-1} - n\cdot(-1)^n \\
&\ \ \vdots \\
&= c_0 - 1\cdot(-1)^1 - 2\cdot(-1)^2 - \cdots - (n-1)\cdot(-1)^{n-1} - n\cdot(-1)^n \\
&= -\sum_{k=1}^{n}(-1)^k k = \sum_{k=0}^{n}(-1)^{k+1}k,
\end{aligned}$$

and this is the first formula of (1.4).
Adding Equations (1.2) and (1.3) (for $n-1$ instead of n), we get the second formula of (1.4).
Finally, let us evaluate $S_n := \sum_{k=0}^{n}(-1)^{k+1}k$. It is well-known that $T_n := \sum_{k=0}^{n} k = \binom{n+1}{2}$. Then $S_n + T_n = \sum_{k=0}^{n}\left(1+(-1)^{k+1}\right)k = 2\sum_{k=1}^{\lfloor\frac{n+1}{2}\rfloor}(2k-1) = 2\left(\left\lfloor\frac{n+1}{2}\right\rfloor\right)^2$. Solving this equation for S_n, we obtain the third formula of (1.4). $\square$

Remark 1.2. By (1.2) we obtain two alternative recurrences for (c_n), namely

$$c_0 = 0,\, c_1 = 1, \quad c_{n+2} = c_n - (-1)^n, \quad n \geq 0, \tag{1.5}$$

and

$$c_0 = 0,\, c_1 = 1,\, c_2 = -1, \quad c_{n+3} = -c_{n+2} + c_{n+1} + c_n, \quad n \geq 0, \tag{1.6}$$

because $c_{n+2} = c_{n+1} - (n+2)(-1)^n = c_n + (n+1)(-1)^n - (n+2)(-1)^n = c_n - (-1)^n$. This proves (1.5). Consequently, $c_{n+3} = c_{n+1} + (-1)^n$, $c_{n+2} = c_n - (-1)^n$. The recurrence relation (1.6) now follows by adding these two equations.

Note that the first formula on the right-hand side of (1.4) states that (c_n) is the sequence of partial sums of $b_n := (-1)^{n+1}n$, $n \in \mathbb{N}_0$, that is $(b_n) = (0, 1, -2, 3, -4, 5, -6, 7, -8, \dots)$ (the sequence A181983) and that the sequence of nonnegative integers $(n) = (0, 1, 2, 3, 4, 5, \dots)$ (the sequence A001477) is given by $((-1)^{n+1}b_n)_{n\in\mathbb{N}_0}$. Hence, in this paper we shall study the hypersequences of $(b_n)_{n\in\mathbb{N}_0}$ and those of the closely related sequence $a_n := (-1)^n$, $n \in \mathbb{N}_0$ (the sequence A033999).

2 Hypersequences of $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$

Let $(f_n)_{n\in\mathbb{N}_0}$ be an arbitrary sequence (of real or complex numbers). Then the *hypersequence of the rth generation* is defined recursively for all $r \in \mathbb{N}$ and $n \in \mathbb{N}_0$ as

$$f_n^{(r)} := \sum_{k=0}^{n} f_k^{(r-1)}, \quad \text{and} \quad f_n^{(0)} := f_n. \tag{2.1}$$

For $r = 1$, we have $f_n^{(1)} = \sum_{k=0}^n f_k^{(0)} = \sum_{k=0}^n f_k$ and this is the sequence of partial sums of $(f_n)_{n\in\mathbb{N}_0}$; for $r = 2$, we have $f_n^{(2)} = \sum_{k=0}^n f_k^{(1)} = \sum_{k=0}^n \left(\sum_{j=0}^k f_j\right)$ and this is the sequence of partial sums of $(f_n^{(1)})_{n\in\mathbb{N}_0}$, and so on.

By means of this definition we obtain the array $(f_n^{(r)})$, where $r \in \mathbb{N}_0$ is the row and $n \in \mathbb{N}_0$ is the column of this array (see Table 1).

The next theorem is well-known (see, e.g., [1, Proposition 2, p. 945] for the special case $f_0^i = f_0$ for all $i \in \{1, \dots, r\}$). The second equation follows from the fact that $k \in \{0, 1, \dots, n\}$ if and only if $n - k \in \{0, 1, \dots, n\}$.

Theorem 2.1. *Let $(f_n)_{n\in\mathbb{N}_0}$ be an arbitrary sequence (of real or complex numbers) and $(f_n^{(r)})_{n\in\mathbb{N}_0}$, $r \in \mathbb{N}_0$, be the hypersequence of the rth generation as defined before. Then for all $r \in \mathbb{N}$ and $n \in \mathbb{N}_0$*

$$f_n^{(r)} = \sum_{k=0}^{n} \binom{n+r-1-k}{r-1} f_k = \sum_{k=0}^{n} \binom{r+k-1}{k} f_{n-k}. \tag{2.2}$$

$r\backslash n$	0	1	2	3	4
0	f_0	f_1	f_2	f_3	f_4
1	f_0	f_0+f_1	$f_0+f_1+f_2$	$f_0+f_1+f_2+f_3$	$f_0+f_1+f_2+f_3+f_4$
2	f_0	$2f_0+f_1$	$3f_0+2f_1+f_2$	$4f_0+3f_1+2f_2+f_3$	$5f_0+4f_1+3f_2+2f_3+f_4$
3	f_0	$3f_0+f_1$	$6f_0+3f_1+f_2$	$10f_0+6f_1+3f_2+f_3$	$15f_0+10f_1+6f_2+3f_3+f_4$
4	f_0	$4f_0+f_1$	$10f_0+4f_1+f_2$	$20f_0+10f_1+4f_2+f_3$	$35f_0+20f_1+10f_2+4f_3+f_4$
5	f_0	$5f_0+f_1$	$15f_0+5f_1+f_2$	$35f_0+15f_1+5f_2+f_3$	$70f_0+35f_1+15f_2+5f_3+f_4$

Table 6: The hypersequences $(f_n^{(r)})_{n\in\mathbb{N}_0}, r \in \mathbb{N}_0$, of $(f_n)_{n\in\mathbb{N}_0}$

Applying this theorem to the sequences $f_n := a_n$ and $f_n := b_n$, $n \in \mathbb{N}_0$, we obtain the following results (see Table 2 and Table 3).

Corollary 2.2. *For all* $r, n \in \mathbb{N}_0$:

$$a_n^{(r)} = \sum_{k=0}^{n} \binom{n+r-1-k}{r-1} (-1)^k = \sum_{k=0}^{n} \binom{r+k-1}{k} (-1)^{n-k} \tag{2.3}$$

$$b_n^{(r)} = \sum_{k=0}^{n} \binom{n+r-1-k}{r-1} (-1)^{k+1} k = \sum_{k=0}^{n} \binom{r+k-1}{k} (-1)^{n-k+1} (n-k). \tag{2.4}$$

The next corollary gives the relationship between these hypersequences.

Corollary 2.3. *For all* $r, n \in \mathbb{N}_0$:

$$a_n^{(r)} = b_n^{(r)} + b_{n+1}^{(r)} \tag{2.5}$$

and, conversely,

$$b_n^{(r)} = \sum_{k=1}^{n} (-1)^{k+1} a_{n-k}^{(r)} = \sum_{k=0}^{n-1} (-1)^{n-k+1} a_k^{(r)}. \tag{2.6}$$

Proof. By (2.4) we have

$$\begin{aligned}
b_{n+1}^{(r)} &= \sum_{k=0}^{n+1}\binom{r+k-1}{k}(-1)^{n+2-k}(n+1-k)\\
&= \sum_{k=0}^{n+1}\binom{r+k-1}{k}(-1)^{n-k}(n-k)+\sum_{k=0}^{n+1}\binom{r+k-1}{k}(-1)^{n-k}\\
&= -\sum_{k=0}^{n}\binom{r+k-1}{k}(-1)^{n-k+1}(n-k)+\binom{r+n}{n+1}+\\
&\quad+\sum_{k=0}^{n}\binom{r+k-1}{k}(-1)^{n-k}-\binom{r+n}{n+1}\\
&= -b_n^{(r)}+a_n^{(r)},
\end{aligned}$$

and this proves (2.5).
Conversely, applying the method of backward substitution we obtain from (2.5)

$$\begin{aligned}
b_n^{(r)} &= -b_{n-1}^{(r)}+a_{n-1}^{(r)}\\
&= -\big(-b_{n-2}^{(r)}+a_{n-2}^{(r)}\big)+a_{n-1}^{(r)} = b_{n-2}^{(r)}-a_{n-2}^{(r)}+a_{n-1}^{(r)}\\
&= -b_{n-3}^{(r)}+a_{n-3}^{(r)}-a_{n-2}^{(r)}+a_{n-1}^{(r)}\\
&= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&= (-1)^n b_0^{(r)}+\sum_{k=1}^{n}(-1)^{k+1}a_{n-k}^{(r)}.
\end{aligned}$$

Together with $b_0^{(r)}=0$ this proves the second formula of (2.6). Finally, the first formula of (2.6) follows from the fact that $k\in\{1,2,\dots,n\}$ if and only if $n-k\in\{0,1,\dots,n-1\}$. $\square$

This corollary shows that knowing $\big(b_n^{(r)}\big)_{n\in\mathbb{N}_0}$, we obtain $\big(a_n^{(r)}\big)_{n\in\mathbb{N}_0}$ from (2.5) and, conversely, knowing $\big(a_n^{(r)}\big)_{n\in\mathbb{N}_0}$, we obtain $\big(b_n^{(r)}\big)_{n\in\mathbb{N}_0}$ from (2.6). The hypersequences of the rth generation $\big(a_n^{(r)}\big)$ and $\big(b_n^{(r)}\big)$ satisfy the following recurrences:

Theorem 2.4. *For all $r\in\mathbb{N}_0$:*

$$a_0^{(r)}=1,\quad a_{n+1}^{(r)}=-a_n^{(r)}+\binom{r+n}{n+1},\quad n\geq 0, \tag{2.7}$$

$$b_0^{(r)}=0,\quad b_1^{(r)}=1,\quad b_{n+1}^{(r)}=-2b_n^{(r)}-b_{n-1}^{(r)}+\binom{r+n-1}{n},\quad n\geq 1. \tag{2.8}$$

$r\backslash n$	0	1	2	3	4	5	6	7	8	9	10
0	1	-1	1	-1	1	-1	1	-1	1	-1	1
1	1	0	1	0	1	0	1	0	1	0	1
2	1	1	2	2	3	3	4	4	5	5	6
3	1	2	4	6	9	12	16	20	25	30	36
4	1	3	7	13	22	34	50	70	95	125	161
5	1	4	11	24	46	80	130	200	295	420	581
6	1	5	16	40	86	166	296	496	791	1211	1792
7	1	6	22	62	148	314	610	1106	1897	3108	4900

Table 7: The hypersequences of $(a_n)_{n\in\mathbb{N}_0}$

$r\backslash n$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	-2	3	-4	5	-6	7	-8	9	-10
1	0	1	-1	2	-2	3	-3	4	-4	5	-5
2	0	1	0	2	0	3	0	4	0	5	0
3	0	1	1	3	3	6	6	10	10	15	15
4	0	1	2	5	8	14	20	30	40	55	70
5	0	1	3	8	16	30	50	80	120	175	245
6	0	1	4	12	28	58	108	188	308	483	728
7	0	1	5	17	45	103	211	399	707	1190	1918

Table 8: The hypersequences of $(b_n)_{n\in\mathbb{N}_0}$

Proof. By (2.4) it follows that for all $n \geq 0$

$$
\begin{aligned}
a_{n+1}^{(r)} &= \sum_{k=0}^{n+1} \binom{r+k-1}{k} (-1)^{n+1-k} \\
&= -\sum_{k=0}^{n} \binom{r+k-1}{k} (-1)^{n-k} + \binom{r+n}{n+1} \\
&= -a_n^{(r)} + \binom{r+n}{n+1}.
\end{aligned}
$$

Together with $a_0^{(r)} = 1$ this proves the assertion (2.7).
By (2.5) and (2.7) it follows that $b_n^{(r)} = -b_{n-1}^{(r)} + a_{n-1}^{(r)}$ and $a_{n-1}^{(r)} = -a_n^{(r)} + \binom{r+n-1}{n}$ for all $n \geq 1$. Substituting the last equation into the first one, we get $b_n^{(r)} = -b_{n-1}^{(r)} - a_n^{(r)} + \binom{r+n-1}{n}$. By (2.5) it follows that $b_n^{(r)} = -b_{n-1}^{(r)} - (b_n^{(r)} + b_{n+1}^{(r)}) + \binom{r+n-1}{n}$. Solving this equation for $b_{n+1}^{(r)}$, we obtain (2.8) valid for all $n \geq 1$. The initial values for all $r \geq 0$ are by definition $b_0^{(r)} = 0$ and $b_1^{(r)} = \sum_{k=0}^{1} b_k^{(r-1)} = b_0^{(r-1)} + b_1^{(r-1)} = b_1^{(r-1)} = b_1^{(r-2)} = \cdots = b_1^{(0)} = 1$, and this proves (2.8). $\square$

We now derive alternative representations of $a_n^{(r)}$ and $b_n^{(r)}$.

Theorem 2.5. *For all $n \in \mathbb{N}_0$:*

$$
a_n^{(0)} = (-1)^n, \quad 2a_n^{(r+1)} = a_n^{(r)} + \binom{r+n}{n}, \quad r \geq 0, \tag{2.9}
$$

with the solution

$$
a_n^{(r)} = \frac{1}{2^r}\left((-1)^n + \sum_{k=0}^{r-1} 2^k \binom{n+k}{k}\right), \tag{2.10}
$$

and for $r \geq 1$

$$
\begin{aligned}
b_n^{(0)} &= (-1)^{n+1} n, \\
b_n^{(1)} &= \frac{1}{4}(1 - (2n+1)(-1)^n), \\
4b_n^{(r+1)} &= 4b_n^{(r)} - b_n^{(r-1)} + \binom{r+n-1}{n-1},
\end{aligned} \tag{2.11}
$$

with the solution

$$
b_n^{(r)} = \sum_{k=0}^{n-1} (-1)^{n-k+1} a_k^{(r)} = \frac{(-1)^{n+1}}{2^r} \cdot \left(n + \sum_{j=0}^{r-1} 2^j \Big(\sum_{k=0}^{n-1} (-1)^k \binom{k+j}{j} \Big)\right). \tag{2.12}
$$

Proof. First, we derive a recurrence relation of $a_n^{(r)}$ with respect to r. By definition, we have $a_n^{(0)} = a_n = (-1)^n$. By (2.3) and by the addition formula for binomial coefficients ([2, Equation (5.8)]) it follows that for all $r \geq 0$

$$\begin{aligned} a_n^{(r+1)} &= \sum_{k=0}^{n} \binom{r+1+k-1}{k} (-1)^{n-k} \\ &= \sum_{k=0}^{n} \binom{r+k-1}{k} (-1)^{n-k} + (-1)^n \sum_{k=0}^{n} \binom{r+k-1}{k-1} (-1)^{n-k}. \end{aligned}$$

The first term on the right-hand side is by definition equal to $a_n^{(r)}$, whereas the second term is equal to

$$\begin{aligned} \sum_{k=1}^{n} \binom{r+k-1}{k-1} (-1)^{n-k} &= \sum_{k=0}^{n-1} \binom{r+k}{k} (-1)^{n-(k+1)} \\ &= -\sum_{k=0}^{n-1} \binom{r+1+k-1}{k} (-1)^{n-k}. \end{aligned}$$

Hence,

$$\begin{aligned} a_n^{(r+1)} &= a_n^{(r)} - \sum_{k=0}^{n-1} \binom{r+1+k-1}{k} (-1)^{n-k} \\ &= a_n^{(r)} - \sum_{k=0}^{n} \binom{r+1+k-1}{k} (-1)^{n-k} + \binom{r+n}{n} \\ &= a_n^{(r)} - a_n^{(r+1)} + \binom{r+n}{n}. \end{aligned}$$

By solving this equation for $a_n^{(r+1)}$, the statement (2.9) follows.
The solution of this recurrence relation can be obtained by the method of backward substitution. However, we simply check that the right-hand side of equation (2.10) satisfies (2.9). For $r = 0$ we get $(-1)^n$. Furthermore,

$$\begin{aligned} 2a_n^{(r+1)} &= 2 \cdot \frac{1}{2^{r+1}} \left((-1)^n + \sum_{k=0}^{r} 2^k \binom{n+k}{k} \right) \\ &= \frac{1}{2^r} \left((-1)^n + \sum_{k=0}^{r-1} 2^k \binom{n+k}{k} + 2^r \binom{n+r}{r} \right) \\ &= \frac{1}{2^r} \left((-1)^n + \sum_{k=0}^{r-1} 2^k \binom{n+k}{k} \right) + \binom{n+r}{r} \\ &= a_n^{(r)} + \binom{r+n}{n} \end{aligned}$$

by the symmetry property of the binomial coefficients. This proves the formula (2.10).
By (2.4) and by the addition formula for binomial coefficients it follows that for all $r \geq 0$

$$\begin{aligned} b_n^{(r+1)} &= \sum_{k=0}^{n} \binom{r+1+k-1}{k} (-1)^{n-k+1}(n-k) \\ &= \sum_{k=0}^{n} \binom{r+k-1}{k} (-1)^{n-k+1}(n-k) + \\ &\quad + \sum_{k=0}^{n} \binom{r+k-1}{k-1} (-1)^{n-k+1}(n-k), \end{aligned}$$

where the first term on the right-hand side is by definition equal to $b_n^{(r)}$, and the second term for k instead of $k-1$ is equal to

$$\begin{aligned} &\sum_{k=1}^{n} \binom{r+k-1}{k-1} (-1)^{n-k+1}(n-k) = \\ &= \sum_{k=0}^{n-1} \binom{r+k}{k} (-1)^{n-(k+1)+1}(n-(k+1)) \\ &= -\sum_{k=0}^{n-1} \binom{r+k}{k} (-1)^{n-k+1}(n-k) - \sum_{k=0}^{n-1} \binom{r+k}{k} (-1)^{n-k}. \end{aligned}$$

The first sum on the right-hand side is by definition equal to

$$-\sum_{k=0}^{n-1} \binom{r+k}{k} (-1)^{n-k+1}(n-k) = -\sum_{k=0}^{n} \binom{r+k}{k} (-1)^{n-k+1}(n-k) = b_n^{(r+1)},$$

whereas the second sum is equal to

$$\begin{aligned} -\sum_{k=0}^{n-1} \binom{r+k}{k} (-1)^{n-k} &= -\sum_{k=0}^{n} \binom{r+k}{k} (-1)^{n-k} + \binom{r+n}{n} \\ &= -a_n^{(r+1)} + \binom{r+n}{n}. \end{aligned}$$

Hence, we obtain

$$b_n^{(r+1)} = b_n^{(r)} - b_n^{(r+1)} - a_n^{(r+1)} + \binom{r+n}{n}$$

and solving for $b_n^{(r+1)}$

$$2b_n^{(r+1)} = b_n^{(r)} - a_n^{(r+1)} + \binom{r+n}{n}, \tag{2.13}$$

or, for $r-1$ instead of r

$$2b_n^{(r)} = b_n^{(r-1)} - a_n^{(r)} + \binom{r-1+n}{n}. \tag{2.14}$$

Subtracting Equation (2.14) from the double of Equation (2.13) we obtain

$$4b_n^{(r+1)} - 2b_n^{(r)} = 2b_n^{(r)} - b_n^{(r-1)} - 2a_n^{(r+1)} + a_n^{(r)} + 2\binom{r+n}{n} - \binom{r-1+n}{n}.$$

Finally, solving for $b_n^{(r+1)}$ and using the recurrence relation (2.9) and the addition formula for binomial coefficients, we obtain

$$\begin{aligned} 4b_n^{(r+1)} &= 4b_n^{(r)} - b_n^{(r-1)} - a_n^{(r)} - \binom{r+n}{n} + a_n^{(r)} + 2\binom{r+n}{n} - \binom{r-1+n}{n} \\ &= 4b_n^{(r)} - b_n^{(r-1)} + \binom{r+n-1}{n-1}. \end{aligned}$$

This recurrence relation is linear and of second order and has the initial values $b_n^{(0)} = (-1)^{n+1}n$ and by Lemma (1.1) $b_n^{(1)} = c_n = \frac{1}{4}(1-(2n+1)(-1)^n)$. This proves the assertion (2.11).

We now show that $g(r,n) := \sum_{k=0}^{n-1}(-1)^{n-k+1}a_k^{(r)}$ solves the recurrence (2.11). First, we have $g(0,n) = \sum_{k=0}^{n-1}(-1)^{n-k+1}a_k^{(0)} = \sum_{k=0}^{n-1}(-1)^{n-k+1}(-1)^k = (-1)^{n+1}n$ and since by (2.10) $a_n^{(1)} = \sum_{k=0}^{n}(-1)^k = \frac{1}{2}(1+(-1)^n)$ it follows that

$$\begin{aligned} g(1,n) &= \sum_{k=0}^{n-1}(-1)^{n-k+1}a_k^{(1)} = \frac{1}{2}(-1)^{n+1}\sum_{k=0}^{n-1}((-1)^k+1) \\ &= \frac{1}{2}(-1)^{n+1}\left(\frac{1}{2}(1+(-1)^{n-1})+n\right) = \frac{1}{4}(1-(2n+1)(-1)^n). \end{aligned}$$

Consequently, $g(r,n)$ satisfies the two initial conditions. Furthermore, by

(2.9) and by the addition formula of the binomial coefficients, we have

$$
\begin{aligned}
&4g(r+1,n) - 4g(r,n) + g(r-1,n) = \\
&= \sum_{k=0}^{n-1} (-1)^{n-k+1} (4a_k^{(r+1)} - 4a_k^{(r)} + a_k^{(r-1)}) \\
&= \sum_{k=0}^{n-1} (-1)^{n-k+1} \Big(2 \cdot (2a_k^{(r+1)} - a_k^{(r)}) - (2a_k^{(r)} - a_k^{(r-1)})\Big) \\
&= \sum_{k=0}^{n-1} (-1)^{n-k+1} \left(2\binom{r+k}{k} - \binom{r-1+k}{k}\right) \\
&= \sum_{k=0}^{n-1} (-1)^{n-k+1} \binom{r+k}{k} + \sum_{k=0}^{n-1} (-1)^{n-k+1} \binom{r+k-1}{k-1}
\end{aligned}
$$

and the right-hand side is equal to $\binom{r+n-1}{n-1}$, since with k instead of $k-1$ the second sum can be expressed as follows:

$$
\begin{aligned}
\sum_{k=0}^{n-1} (-1)^{n-k+1} \binom{r+k-1}{k-1} &= \sum_{k=0}^{n-2} (-1)^{n-k} \binom{r+k}{k} \\
&= -\sum_{k=0}^{n-2} (-1)^{n-k+1} \binom{r+k}{k}.
\end{aligned}
$$

It follows that $g(r,n) = b_n^{(r)}$. Furthermore, by (2.10) it follows that

$$
\begin{aligned}
b_n^{(r)} &= \sum_{k=0}^{n-1} (-1)^{n-k+1} a_k^{(r)} = \sum_{k=0}^{n-1} (-1)^{n-k+1} \cdot \frac{1}{2^r} \left((-1)^k + \sum_{j=0}^{r-1} 2^j \binom{k+j}{j}\right) \\
&= \frac{1}{2^r} \sum_{k=0}^{n-1} (-1)^{n+1} + \frac{1}{2^r} \sum_{k=0}^{n-1} (-1)^{n-k+1} \sum_{j=0}^{r-1} 2^j \binom{k+j}{j} \\
&= \frac{(-1)^{n+1} n}{2^r} + \frac{(-1)^{n+1}}{2^r} \sum_{k=0}^{n-1} (-1)^k \sum_{j=0}^{r-1} 2^j \binom{k+j}{j} \\
&= \frac{(-1)^{n+1}}{2^r} \left(n + \sum_{j=0}^{r-1} 2^j \Big(\sum_{k=0}^{n-1} (-1)^k \binom{k+j}{j}\Big)\right),
\end{aligned}
$$

and this proves (2.12). $\square$

Note that by (2.3) and (2.10) we have shown that for all r, $n \in \mathbb{N}_0$, we have

$$
\sum_{k=0}^{n} \binom{r+k-1}{k} (-1)^{n-k} = \frac{1}{2^r} \left((-1)^n + \sum_{k=0}^{r-1} 2^k \binom{n+k}{k}\right), \tag{2.15}
$$

and by (2.4) and (2.12) we have

$$\begin{aligned}&\sum_{k=0}^{n}\binom{r+k-1}{k}(-1)^{n-k+1}(n-k)=\\&=\frac{(-1)^{n+1}}{2^r}\Bigl(n+\sum_{j=0}^{r-1}2^j\bigl(\sum_{k=0}^{n-1}(-1)^k\binom{k+j}{j}\bigr)\Bigr).\end{aligned}\tag{2.16}$$

By (2.10) the first few values of $a_n^{(r)}$ for fixed $r \geq 0$ are:

i) $r = 0$: $a_n^{(0)} = (-1)^n$, (A033999)

ii) $r = 1$: $a_n^{(1)} = \frac{1}{2}(1 + (-1)^n)$, (A059841, characteristic function of even numbers)

iii) $r = 2$: $a_n^{(2)} = \frac{1}{4}(1+(-1)^n)+\frac{1}{2}(n+1)$, (A004526$(n+1)$, nonnegative integers repeated)

iv) $r = 3$: $a_n^{(3)} = \frac{1}{8}(1 + (-1)^n) + \frac{1}{4}(n + 1)(n + 3)$, (A002620$(n + 2)$)

v) $r = 4$: $a_n^{(4)} = \frac{1}{16}(1 + (-1)^n) + \frac{1}{24}(n + 1)(n + 3)(2n + 7)$, (A002623)

vi) $r = 5$: $a_n^{(5)} = \frac{1}{32}(1 + (-1)^n) + \frac{1}{48}(n + 1)(n + 3)^2(n + 5)$, (A001752).

In particular, setting $r = 3$ and $n = 2m$, $m \in \mathbb{N}_0$, in (2.15) gives the sequence of the square numbers A000290$(m + 1)$

$$\sum_{k=0}^{2m}\binom{k+2}{2}(-1)^k = (m+1)^2,$$

while for $n = 2m + 1$, $m \in \mathbb{N}_0$, we get the sequence of the oblong numbers A002378$(m + 1)$

$$\sum_{k=0}^{2m+1}\binom{k+2}{2}(-1)^{k-1} = (m+1)(m+2).$$

On the other hand, again by (2.10) the first few sequences of $a_n^{(r)}$ for fixed $n \geq 0$ are:

i) $n = 0$: $a_0^{(r)} = 1$, (A000012, the all 1's sequence)

ii) $n = 1$: $a_1^{(r)} = r - 1$, (A023443)

iii) $n = 2$: $a_2^{(r)} = \frac{1}{2}(r^2 - r + 2)$, (A152947)

iv) $n = 3$: $a_3^{(r)} = \frac{1}{6}(r^3 + 5r - 6)$, (A283551)

v) $n = 4$: $a_4^{(r)} = \frac{1}{24}(r^4 + 2r^3 + 11r^2 - 14r + 24)$, (after removing the first term this is the sequence A223718)

vi) $n = 5$: $a_5^{(r)} = \frac{1}{120}(r^5 + 5r^4 + 25r^3 - 5r^2 + 94r - 120)$, (after removing the first two terms, this is the sequence A257890).

Remark 2.6. A look at the polynomials $n! \cdot a_n^{(r)}$ (for fixed $n \geq 0$) shows that their coefficients in descending powers of r are given by the triangle as shown in Table 4. This table is up to the sign of the columns for odd k given by the triangle A054651. By a slight modification of the formula given in A054651, these coefficients $U(n,k)$, $n, k \in \mathbb{N}_0$, are given by

$$U(n,k) = (-1)^k \sum_{i=0}^{k} \begin{bmatrix} i+n-k \\ n-k \end{bmatrix} \frac{n!}{(i+n-k)!}, \tag{2.17}$$

where $\left[\begin{smallmatrix} i+n-k \\ n-k \end{smallmatrix}\right]$ is an unsigned Stirling number of the first kind. Hence, by (2.3) we have

$$\begin{aligned}\sum_{k=0}^{n} \binom{r+k-1}{k} (-1)^{n-k} &= \frac{1}{n!} \sum_{k=0}^{n} U(n,k) r^{n-k} \\ &= \sum_{k=0}^{n} \left((-1)^k \sum_{i=0}^{k} \begin{bmatrix} i+n-k \\ n-k \end{bmatrix} \frac{1}{(i+n-k)!} \right) r^{n-k}.\end{aligned} \tag{2.18}$$

Note that $U(n,n) = (-1)^n \cdot n!$ with the first few values $(1, -1, 2, -6, 24, -120, 720, -5040, \ldots)$ (A133942) and that the sequence of the row sums is $S_1(n) = \sum_{k=0}^{n} U(n,k) = \frac{1+(-1)^n}{2} \cdot n!$ (A005359) with the first few values $(1, 0, 2, 0, 24, 0, 720, 0, \ldots)$, while the sequence of the alternating row sums given by $S_2(n) = \sum_{k=0}^{n} (-1)^k U(n,k)$ with the first few values $(1, 2, 4, 12, 52, 250, 1608, 10808, \ldots)$ is not in the OEIS.

By (2.12) the first few values of $b_n^{(r)}$ for fixed $r \geq 0$ are:

i) $r = 0$: $b_n^{(0)} = (-1)^{n+1} n$ (A181983)

ii) $r = 1$: $b_n^{(1)} = \frac{1}{4}(1 - (2n+1)(-1)^n)$ (A001057, canonical enumeration of integers)

iii) $r = 2$: $b_n^{(2)} = \frac{1}{4}(n+1)(1 - (-1)^n)$ (A142150$(n+1)$, the nonnegative integers interleaved with 0's)

$n\backslash k$	0	1	2	3	4	5	6	7
0	1							
1	1	-1						
2	1	-1	2					
3	1	0	5	-6				
4	1	2	11	-14	24			
5	1	5	25	-5	94	-120		
6	1	9	55	75	304	-444	720	
7	1	14	112	350	1099	-364	3828	-5040

Table 9: Triangle $U(n,k)$ of the coefficients (in descending powers of r) of the polynomials $n!\cdot a_n^{(r)}$

iv) $r=3$: $b_n^{(3)} = \frac{1}{16}(2n^2+6n+3-(2n+3)(-1)^n)$ (removing the first term this is the sequence A008805$(n-1)$, $n\geq 1$, triangular numbers repeated)

v) $r=4$: $b_n^{(4)} = \frac{1}{48}(2n^3+12n^2+19n+6-3(n+2)(-1)^n)$ (A006918)

vi) $r=5$: $b_n^{(5)} = \frac{1}{192}(2n^4+20n^3+64n^2+70n+15-3(2n+5)(-1)^n)$ (removing the first term this is the sequence A002624$(n-1)$, $n\geq 1$).

In particular, setting $r=3$ and $n=2m$, $m\in\mathbb{N}_0$, in (2.16) gives the sequence of the triangular numbers A000217

$$\sum_{k=0}^{2m}\binom{k+2}{2}(-1)^{k-1}(2m-k) = \frac{1}{2}m(m+1) = \binom{m+1}{2},$$

while for $n=2m+1$, $m\in\mathbb{N}_0$, we also get the sequence of the triangular numbers A000217$(m+1)$

$$\sum_{k=0}^{2m+1}\binom{k+2}{2}(-1)^{k-1}(2m+1-k) = \frac{1}{2}(m+1)(m+2) = \binom{m+2}{2}.$$

On the other hand, again by (2.12) the first few sequences of $b_n^{(r)}$ for fixed $n\geq 0$ are:

i) $n=0$: $b_0^{(r)} = 0$, (A000004, the zero sequence)

ii) $n=1$: $b_1^{(r)} = 1$, (A000012, the all 1's sequence)

iii) $n=2$: $b_2^{(r)} = r-2$, (A023444)

$n\backslash k$	0	1	2	3	4	5	6
0	0						
1	1						
2	1	-2					
3	1	-3	6				
4	1	-3	14	-24			
5	1	-2	23	-70	120		
6	1	0	35	-120	444	-720	
7	1	3	55	-135	1024	-3108	5040

Table 10: Triangle $V(n,k)$ of the coefficients (in descending powers of r) of the polynomials $(n-1)! \cdot b_n^{(r)}$.

iv) $n=3$: $b_3^{(r)} = \frac{1}{2}(r^2 - 3r + 6)$, (after removing the first term this is the sequence A152948)

v) $n=4$: $b_4^{(r)} = \frac{1}{6}(r^3 - 3r^2 + 14r - 24)$, (this sequence is not in the OEIS)

vi) $n=5$: $b_5^{(r)} = \frac{1}{24}(r^4 - 2r^3 + 23r^2 - 70r + 120)$, (this sequence is not in the OEIS).

Remark 2.7. A look at the polynomials $(n-1)! \cdot b_n^{(r)}$ (for fixed $n \geq 0$) shows that their coefficients $V(n,k)$, $n,k \in \mathbb{N}_0$, in descending powers of r are given by the triangle as shown in Table 5. Note also that $V(n,n-1) = (-1)^{n+1} \cdot n!$, $n \geq 1$, (sequence A155456(n+2)) with the first few values $(1,-2,6,-24,120,-720,5040,\ldots)$ and that the sequence of the row sums is $T_1(n) = \sum_{k=0}^{n-1} V(n,k) = (-1)^{n+1} n! \cdot \frac{2n+3}{4}$, $n \geq 1$, (after removing the first term, the unsigned sequence $|T_1(n)|$ is A052558) with the first few values $(0,1,-1,4,-12,72,-360,2880,\ldots)$, while the sequence of the alternating row sums given by $T_2(n) = \sum_{k=0}^{n-1}(-1)^k V(n,k)$ with the first few values $(0,1,3,10,42,216,1320,9366,\ldots)$ is not in the OEIS.

3 Generating functions and two special subsequences

We now determine the ordinary generating functions for $(a_n^{(r)})_{n\in\mathbb{N}_0}$ and $(b_n^{(r)})_{n\in\mathbb{N}_0}$, denoted by $F_r(s)$ and $G_r(s)$, respectively. We recall that the ordinary generating function for the sequence $(f_n)_{n\in\mathbb{N}_0}$ is defined as the (formal) power series $\sum_{n=0}^{\infty} f_n s^n$.

Proposition 3.1. *The ordinary generating function for* $a_n^{(r)}$ *is given by*

$$F_r(s) = \frac{1}{1+s} \cdot \frac{1}{(1-s)^r}, \tag{3.1}$$

and that for $b_n^{(r)}$ *is given by*

$$G_r(s) = \frac{s}{(1+s)^2} \cdot \frac{1}{(1-s)^r} = \frac{s}{1+s} \cdot F_r(s). \tag{3.2}$$

Proof. By definition and by [2, Equation (7.21)], we have for all $r \geq 1$:

$$F_r(s) = \sum_{n=0}^{\infty} a_n^{(r)} s^n = \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} a_k^{(r-1)} \Big) s^n = \frac{1}{1-s} F_{r-1}(s).$$

The solution of this recurrence relation is given by $F_r(s) = F_0(s) \cdot \frac{1}{(1-s)^r}$. Since the generating function $F_0(s)$ for $a_n = (-1)^n$ is given by the geometric series $F_0(s) = \sum_{n=0}^{\infty} (-1)^n s^n = \frac{1}{1-(-s)} = \frac{1}{1+s}$, the assertion (3.1) is proved. Similarly, by definition and by [2, Equation (7.21)], we have for all $r \geq 1$:

$$G_r(s) = \sum_{n=0}^{\infty} b_n^{(r)} s^n = \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} b_k^{(r-1)} \Big) s^n = \frac{1}{1-s} G_{r-1}(s)$$

which has the solution $G_r(s) = G_0(s) \cdot \frac{1}{(1-s)^r}$. The generating function $G_0(s)$ for $b_n = (-1)^{n+1} n$ can be determined in the following way

$$\begin{aligned} G_0(s) &= \sum_{n=0}^{\infty} (-1)^{n+1} n s^n = \sum_{n=0}^{\infty} (-1)^{n+1} (n+1-1) s^n \\ &= \frac{1}{s} \sum_{n=0}^{\infty} (-1)^{n+1} (n+1) s^{n+1} - \sum_{n=0}^{\infty} (-1)^{n+1} s^n \\ &= \frac{1}{s} \sum_{n=0}^{\infty} (-1)^n n s^n + \sum_{n=0}^{\infty} (-1)^n s^n = \frac{1}{s} G_0(s) + \frac{1}{1+s}. \end{aligned}$$

Solving for $G_0(s)$, we get the formula (3.2). $\square$

Finally, we consider the two subsequences $(d_n)_{n \in \mathbb{N}_0}$ and $(e_n)_{n \in \mathbb{N}_0}$, defined as $d_n := a_n^{(n)}$ and $e_n := b_n^{(n)}$, $n \geq 0$, which form the main diagonal of the arrays $(a_n^{(r)})$ and $(b_n^{(r)})$, respectively (see Table 2 and Table 3). We recall that $(C(n))_{n \in \mathbb{N}_0}$ is the sequence of the Catalan numbers, the sequence A000108, defined by $C(n) := \frac{1}{n+1} \binom{2n}{n}$, $n \geq 0$.

Proposition 3.2. *The sequences* $(d_n)_{n\in\mathbb{N}_0}$ *and* $(e_n)_{n\in\mathbb{N}_0}$ *satisfy the recurrences*

$$d_0 = 1, \quad 2d_{n+1} = -d_n + (3n+1)C(n), \quad n \geq 0, \tag{3.3}$$

and

$$e_0 = 0, \quad e_1 = 1, \quad 4e_{n+1} + 4e_n + e_{n-1} = (9n-5)C(n-1), \quad n \geq 1. \tag{3.4}$$

The solutions are $d_n = \sum_{k=0}^{n} \binom{n+k-1}{k}(-1)^{n-k}$ *and* $e_n = \sum_{k=0}^{n} \binom{n+k-1}{k}(-1)^{n-k+1}(n-k)$, $n \geq 0$.

Proof. By definition, we have $d_0 = a_0^{(0)} = 1$. By (2.7) for $r = n+1$ it follows that

$$d_{n+1} = a_{n+1}^{(n+1)} = -a_n^{(n+1)} + \binom{2n+1}{n+1}. \tag{3.5}$$

Furthermore, by (2.9) for $r = n$, we have

$$2a_n^{(n+1)} = a_n^{(n)} + \binom{2n}{n} = d_n + \binom{2n}{n}.$$

Hence, substituting this equation into (3.5) multiplied by 2, we get

$$2d_{n+1} = -2a_n^{(n+1)} + 2\binom{2n+1}{n+1} = -d_n - \binom{2n}{n} + 2\binom{2n+1}{n},$$

and this is the recurrence (3.3), since $2\binom{2n+1}{n} - \binom{2n}{n} = \binom{2n}{n}(2\frac{2n+1}{n+1} - 1) = \frac{3n+1}{n+1}\binom{2n}{n} = (3n+1)C(n)$. The solution of the recurrence (3.3) is given by (2.3) for $r = n$.
By definition, we have $e_0 = b_0^{(0)} = 0$ and $e_1 = b_1^{(1)} = 1$. For $r = n+1$, we get from (2.8)

$$e_{n+1} = b_{n+1}^{(n+1)} = -2b_n^{(n+1)} - b_{n-1}^{(n+1)} + \binom{2n}{n}, \quad n \geq 1. \tag{3.6}$$

For $r = n$, we get from (2.11)

$$4b_n^{(n+1)} = 4b_n^{(n)} - b_n^{(n-1)} + \binom{2n-1}{n-1} = 4e_n - b_n^{(n-1)} + \frac{1}{2}\binom{2n}{n}, \quad n \geq 1, \tag{3.7}$$

and for $r = n$ and $n-1$ instead of n

$$4b_{n-1}^{(n+1)} = 4b_{n-1}^{(n)} - b_{n-1}^{(n-1)} + \binom{2n-2}{n-2} = 4b_{n-1}^{(n)} - e_{n-1} + \binom{2n-2}{n}, \quad n \geq 1. \tag{3.8}$$

Furthermore, by definition of the hypersequence, we have

$$e_n = b_n^{(n)} = b_{n-1}^{(n)} + b_n^{(n-1)}, \quad n \geq 1, \tag{3.9}$$

and

$$b_n^{(n+1)} = b_{n-1}^{(n+1)} + b_n^{(n)} = b_{n-1}^{(n+1)} + e_n, \quad n \geq 1. \tag{3.10}$$

Setting $b_{n-1}^{(n)} = \alpha$, $b_{n-1}^{(n+1)} = \beta$, $b_n^{(n-1)} = \gamma$, and $b_n^{(n+1)} = \delta$, we obtain from (3.6), (3.7), (3.8), (3.9) and (3.10) the following linear system consisting of 5 equations for the 4 unknowns α, β, γ, and δ.

$$\begin{aligned}
e_{n+1} &= -2\delta - \beta + \binom{2n}{n} \\
4\delta &= 4e_n - \gamma + \frac{1}{2}\binom{2n}{n} \\
4\beta &= 4\alpha - e_{n-1} + \binom{2n-2}{n} \\
e_n &= \alpha + \gamma \\
\delta &= \beta + e_n.
\end{aligned}$$

After algebraic elimination of the values α, β, γ, δ, the equation $4e_{n+1} + 4e_n + e_{n-1} = 2\binom{2n}{n} + \binom{2n-2}{n}$ remains, which is (3.4), since $2\binom{2n}{n} + \binom{2n-2}{n} = \frac{(2n-2)!}{(n-1)!(n-1)!} \cdot \left(2\frac{(2n-1)2n}{n \cdot n} + \frac{n-1}{n}\right) = \binom{2(n-1)}{n-1}\frac{9n-5}{n} = (9n-5)C(n-1)$, $n \geq 1$. The solution of the recurrence (3.4) is given by (2.4) for $r = n$. □

The sequence $(d_n)_{n\in\mathbb{N}_0}$ with the first few values $(1, 0, 2, 6, 22, 80, 296, 1106, \ldots)$ is the sequence A072547, while the sequence $(e_n)_{n\in\mathbb{N}_0}$ with the first few values $(0, 1, 0, 3, 8, 30, 108, 399, \ldots)$ is not in the OEIS.

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Normal edge-colorings and superpositions: an overview

Jelena Sedlar and Riste Škrekovski

Abstract

A normal 5-edge-coloring of a cubic graph is a coloring such that for every edge, the number of distinct colors incident to its end-vertices is 3 or 5 (and not 4). The well-known Petersen Coloring Conjecture is equivalent to the statement that every bridgeless cubic graph has a normal 5-edge-coloring. All 3-edge-colorings of a cubic graph are obviously normal, so in order to establish the conjecture, it is sufficient to consider only snarks. The most general known method for constructing snarks is superposition. In this paper, we give an overview of our results on the normal 5-edge-colorings of superpositioned snarks. A family of superpositioned snarks considered here is obtained from a snark G by superpositioning vertices and edges along a cycle C of G by two specific supervertices and by superedges of the form $H_{x,y}$, where H is any snark and x, y a pair of non-adjacent vertices in H. We assume that a snark G has a normal 5-edge-coloring σ and we extend σ to a superpositioned snark $\tilde{G}$. Our consideration starts with superpositions by the Petersen graph P_{10}, where we encounter problems with superpositions along odd cycles. We provide an example of a superposition by P_{10} along an odd cycle C in which σ cannot be extended to a superposition. This does not contradict the Petersen coloring conjecture, since the superposition does have a normal 5-edge-coloring, but not such that it is an extension of σ. We generalize our approach to superpositions by any superedge $H_{x,y}$, where $d(x, y) \geq 3$. For such superpositions, we give two sufficient conditions under which σ can be extended to a superposition. These

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conditions are applied to superpositions by Hypohamiltonian snarks and by Flower snarks, showing thus that some of the former and all of the latter have a normal 5-edge-colorings. Since the Petersen Coloring Conjecture implies some other well-known classical conjectures like the Ford-Fulkerson Conjecture, these results immediately yield some known results on this conjecture.

Keywords: normal edge-coloring; cubic graph; snark; superposition; Petersen Coloring Conjecture.
2020 ***Mathematics Subject Classification***: 05C15.

1 Introduction

A *k-edge-coloring* of a graph G is a function $\sigma : E(G) \to \{1, \dots, k\}$. If an edge-coloring assigns distinct colors to any two adjacent edges in G, it is said to be *proper*. Throughout the paper, we will omit the word 'proper' tacitly assuming properness unless explicitly stated otherwise. For any vertex $v \in V(G)$, the set of colors associated with the edges incident to v is denoted by $\sigma(v)$.

Definition 1.1. Consider a bridgeless cubic graph G, a proper edge-coloring σ, and an edge $uv \in E(G)$. The edge uv is defined as *poor* if $|\sigma(u) \cup \sigma(v)| = 3$, and as *rich* if $|\sigma(u) \cup \sigma(v)| = 5$.

An edge-coloring of a cubic graph G is said to be a *normal edge-coloring* if all edges of G are either poor or rich. This concept was first introduced by Jaeger in [10]. The *normal chromatic index* of G, written as $\chi'_N(G)$, is the minimum value of k for which a normal k-edge-coloring exists. Notably, $\chi'_N(G)$ is always at least 3, and it can never equal 4.
The Petersen Coloring Conjecture is one of the most prominent open problems in graph theory. This conjecture is particularly challenging to prove, as it has been shown to imply several other well-known conjectures, including the Berge-Fulkerson Conjecture and the (5,2)-cycle-cover Conjecture. Interestingly, the Petersen Coloring Conjecture can be reformulated in terms of normal edge-colorings, as noted in [10].

Conjecture 1.2. If G is a bridgeless cubic graph, then $\chi'_N(G) \leq 5$.

It is evident that Conjecture 1.2 holds for every cubic graph G that admits a proper 3-edge-coloring, as such a coloring is a normal edge-coloring where all edges are poor. By Vizing's theorem, every cubic graph is either 3-edge-colorable or 4-edge-colorable. Therefore, to confirm Conjecture 1.2, it suffices to show that it applies to all bridgeless cubic graphs that are not 3-edge-colorable.

Superpositioning snarks. Cubic graphs that are not 3-edge-colorable are commonly referred to as snarks [5, 21]. To exclude trivial cases, the definition of a snark often includes additional conditions related to connectivity. However, these conditions are not crucial for the purposes of this paper. Hence, we adopt a broader definition, considering a *snark* to be any bridgeless cubic graph that is not 3-edge-colorable. Some families of snarks have already been shown to admit normal 5-edge-colorings; see, for instance, [4, 7].

The most general method currently known for generating new snarks from existing ones is the process of superposition [1, 3, 11, 12, 16]. Since this paper explores certain snarks created through superposition, we begin by introducing the method.

Definition 1.3. A *multipole* $M = (V, E, S)$ is defined by a set of vertices $V = V(M)$, a set of edges $E = E(M)$, and a set of semiedges $S = S(M)$. A semiedge is either incident to a single vertex or paired with another semiedge, forming what is known as an *isolated edge* within the multipole.

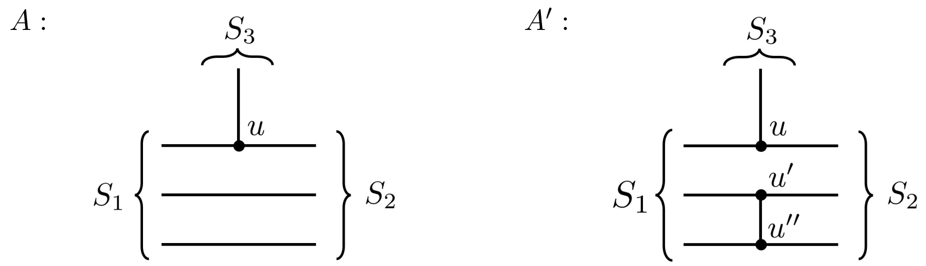

Figure 1: Supervertices A and A', with connectors S_1, S_2, and S_3, where S_1 is a 1-connector, and S_2 and S_3 are 3-connectors.

An example of a multipole can be found in Figure 1. For any vertex v in a multipole M, the *degree* $d_M(v)$ is defined as the total number of edges and semiedges in M that are incident to v. A multipole M is termed *cubic* if every vertex of M has degree 3. For instance, both multipoles depicted in Figure 1 are cubic. Throughout this paper, we focus exclusively on cubic multipoles.

Now, let us introduce some terminology related to the semiedges of a multipole M. A multipole M is referred to as a *k-pole* when the total number of semiedges, $|S(M)|$, equals k. If the set of semiedges S is divided into n subsets S_i such that $|S_i| = k_i$, the multipole is called a *$(k_1, \ldots, k_n)$-pole* and is denoted as $M = (V, E, S_1, \ldots, S_n)$. These subsets S_i are referred to as the *connectors* of M. A connector S_i containing k_i semiedges is specifically called a k_i*-connector*.

Definition 1.4. A *supervertex* (respectively, a *superedge*) is defined as a cubic multipole with three (respectively, two) connectors.

We specifically define the supervertices A and A' as shown in Figure 1. These two supervertices will be the focus of our analysis throughout the paper. Next, we introduce a particular type of superedge relevant to our discussion. Let G be a snark, and let u and v be two non-adjacent vertices in G. The superedge $G_{u,v}$ is constructed by removing the vertices u and v from G, and replacing the three edges incident to u (and similarly for v) with three semiedges in $G_{u,v}$, which collectively form a connector.

Definition 1.5. A *proper superedge* is defined as either an isolated edge or a superedge $G_{u,v}$ where G is a snark.

While the definition of a proper superedge given in [11] is much broader, the simplified definition presented here is sufficient for the scope of this paper. In our work, we will also explore normal edge-colorings of multipoles, as the concept of normal edge-coloring extends naturally to these structures. Let us now formalize this notion.

For a multipole $M = (V, E, S)$, a *(proper) k-edge-coloring* is defined as a function $\sigma : E(M) \cup E(S) \to \{1, \dots, k\}$ such that no two edges or semiedges sharing the same color are incident to the same vertex. Furthermore, a *normal edge-coloring* of a multipole is a proper edge-coloring where every edge is either rich or poor. It is important to note that this definition places no restrictions on the coloring of semiedges.

For a cubic graph $G = (V, E)$, we define two functions: $\mathcal{V}$, which maps each vertex $v \in V$ to a supervertex $\mathcal{V}(v)$, and $\mathcal{E}$, which maps each edge $e \in E$ to a superedge $\mathcal{E}(e)$. A *superposition* $G(\mathcal{V}, \mathcal{E})$ is constructed under the following condition: semiedges of a connector in $\mathcal{V}(v)$ are matched with semiedges of a connector in $\mathcal{E}(e)$ if and only if e is incident to v in G. Naturally, this requires the connectors in $\mathcal{V}(v)$ and $\mathcal{E}(e)$ to have the same number of semiedges.

Observe that the resulting graph $G(\mathcal{V}, \mathcal{E})$ is again cubic. Such a superposition is said to be *proper* if every superedge $\mathcal{E}(e)$ is proper. Additionally, we assume that some vertices and edges of G may be superpositioned by themselves. Formally, such vertices are superpositioned by trivial supervertices consisting of a single vertex with three incident semiedges, and such edges by trivial superedges consisting of a single isolated edge.

The following theorem, as established in [11], applies to snarks of girth ≥ 5 that are cyclically 4-edge connected. However, it is worth noting that the result remains valid for snarks with smaller girths as well.

Theorem 1.6. *For a snark G, any proper superposition $G(\mathcal{V}, \mathcal{E})$ is also a snark.*

Normal colorings of superpositioned snarks. Normal 5-edge-colorings for certain families of superpositioned snarks are analyzed in the series of papers [24–26]. All three papers consider snarks obtained from a snark G by superpositioning the vertices and edges of a cycle C of G by supervertices A or A' and by superedges of the form $H_{x,y}$ where H is any snark and x, y a pair of nonadjacent vertices of H. In all three papers, the same approach is used, where it is assumed that G does have a normal 5-edge-coloring σ and this coloring is then extended to a superposition. Papers [24] and [26] investigate the case when $H = P_{10}$ for every edge of a cycle C. With such a superposition, the problem with the construction of a normal 5-edge-colorings arises when C is an odd-length cycle.
It is established that the problem is not inherent to the approach, as the example of a snark G and its superposition is provided, where the superposition does not have a normal 5-edge-coloring, which is an extension of the coloring of G. Instead, the problem arises due to $H = P_{10}$ being a small snark of the diameter only two. Hence, in [25] the approach is extended to a superopsition by any snark H and any pair of vertices x, y of H with $d(x, y) \geq 3$. Here, two sufficient conditions are given under which a superposition does have a normal 5-edge-coloring are given, the first one is applied to some superpositions by Hypohamiltonian snarks and the other to all superpositions by Flower snarks, showing thus that all these superpositions have a normal 5-edge-coloring, thus the Petersen Coloring Conjecture is verified for them. Since the Petersen Coloring Conjecture implies the Ford-Fulkerson Conjecture, these results immediately yield the results of [14]. In this paper, we give an overview of all these results.

2 Preliminaries

Let G be a snark, and let $C = u_0u_1 \cdots u_{g-1}u_0$ represent a cycle of length g in G. The edges of the cycle C are denoted as $e_i = u_iu_{i+1}$ for $i = 0, \ldots, g-1$, where indices are taken modulo g. Additionally, let v_i denote the neighbor of u_i that is distinct from u_{i-1} and u_{i+1}, and define $f_i = u_iv_i$.
The supervertices A and A' are defined as shown in Figure 1. For superedges, we use $H_{x,y}$, where H is a snark and x, y are two non-adjacent vertices of H. Observe that $H_{x,y}$ contains a pair of 3-connectors, denoted S_x and S_y. These connectors consist of the three semiedges that correspond to halves of the three edges in H incident to x and y, respectively. We now formally define the type of superpositions considered in this paper.

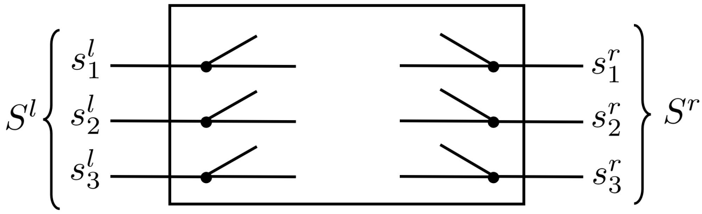

Figure 2: A schematic representation of a superedge $\mathcal{B}_i$ in a superposition $G_C(\mathcal{A}, \mathcal{B})$, showing its left and right connectors along with their semiedges. This representation will be assumed throughout the paper.

Definition 2.1. Let $C = u_0u_1 \cdots u_{g-1}u_0$ be a cycle in a snark G, and let $e_i = u_iu_{i+1}$ represent an edge of C for $i = 0, \dots, g-1$. A *superpositioned snark* $G_C(\mathcal{A}, \mathcal{B})$ is a superposition of G such that:

- For every vertex u_i of C, $\mathcal{A}(u_i) \in \{A, A'\}$.
- For every edge e_i of C, $\mathcal{B}(e_i) \in \{H_{x,y} : H$ is a snark and x, y are non-adjacent vertices of $H\}$.

All other vertices and edges of G are superpositioned by themselves.

Note that the snark H used to construct a superedge $\mathcal{B}_i$ does not need to be the same for different edges of the cycle C. The family of all such superpositions is denoted by $\mathcal{G}_C(\mathcal{A}, \mathcal{B})$. For simplicity, we will write $\mathcal{A}_i$ instead of $\mathcal{A}(u_i)$ and $\mathcal{B}_i$ instead of $\mathcal{B}(e_i)$.
In a superedge $\mathcal{B}_i$, the connector that is matched with a connector of $\mathcal{A}_i$ will be referred to as the *left connector* and denoted by S^l, while the connector matched with $\mathcal{A}_{i+1}$ will be called the *right connector* and denoted by S^r. The three semiedges belonging to the left connector are called the *left semiedges*, labeled as s_1^l, s_2^l, and s_3^l. Similarly, the semiedges of the right connector are the *right semiedges*, denoted as s_1^r, s_2^r, and s_3^r. Figure 2 illustrates the schematic structure of a superedge that will be used consistently throughout this paper.
When a superedge $\mathcal{B}_i$ is derived from a snark H by removing two non-adjacent vertices x and y, it is written as $\mathcal{B}_i = H_{x,y}$ if the left connector corresponds to S_x, and as $\mathcal{B}_i = H_{y,x}$ if the left connector corresponds to S_y. It is useful to assume that the connection between a supervertex $\mathcal{A}_i$ and the superedges $\mathcal{B}_{i-1}$ and $\mathcal{B}_i$ is established as follows: first, the right semiedges of $\mathcal{B}_{i-1}$ are matched with the left semiedges of $\mathcal{B}_i$. Then, one of the resulting edges is subdivided to create the vertex u_i of $\mathcal{A}_i$. Since the identification of

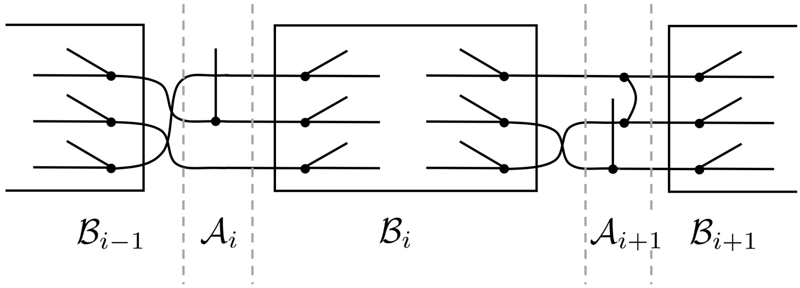

Figure 3: This figure demonstrates how a superedge $\mathcal{B}_i$ is connected to its neighboring superedges $\mathcal{B}_{i-1}$ and $\mathcal{B}_{i+1}$, based on a permutation p_i and a dock d_i associated with $\mathcal{B}_i$. In this example, $p_i = (1, 3, 2)$ and $d_i = 2$. The figure also reveals that $p_{i-1} = (2, 3, 1)$ and $d_{i+1} = 3$.

semiedges between $\mathcal{B}_{i-1}$ and $\mathcal{B}_i$ can occur in multiple ways, as illustrated in Figure 3, we associate a permutation p_{i-1} of the set $\{1, 2, 3\}$ with the superedge $\mathcal{B}_{i-1}$. This permutation specifies how the right semiedges s^r_1, s^r_2, s^r_3 of $\mathcal{B}_{i-1}$ are reordered before being matched with the left semiedges s^l_1, s^l_2, s^l_3 of $\mathcal{B}_i$. Specifically, the semiedge $s^r_{p^{-1}_{i-1}(j)}$ of $\mathcal{B}_{i-1}$ is matched with the semiedge s^l_j of $\mathcal{B}_i$. The permutation p_{i-1} is referred to as a *semiedge permutation*.
For instance, in Figure 3, the semiedge permutations are $p_{i-1} = (2, 3, 1)$ and $p_i = (1, 3, 2)$. When the specific permutation p_{i-1} is clear from the context, we will write $p^{-1}_{i-1}(j)$ simply as j^- for brevity.
Suppose that the edge created by semiedge identification is denoted by $s^r_{j^-}s^l_j$. Among these edges, one $j \in \{1, 2, 3\}$ is selected as the index of the edge to be subdivided to form the vertex u_i. Since j corresponds to a left semiedge in $\mathcal{B}_i$, this choice is represented by $j = d_i$ and is associated with $\mathcal{B}_i$. The value d_i, which determines the left semiedge of $\mathcal{B}_i$ to be connected to the vertex u_i of $\mathcal{A}_i$, is called the *dock* index, and the semiedge $s^l_{d_i}$ is referred to as the *dock* semiedge. For example, in Figure 3, the dock indices are $d_i = 2$ and $d_{i+1} = 3$.
To summarize, each superedge $\mathcal{B}_i$ is associated with a permutation p_i, which determines how the right semiedges of $\mathcal{B}_i$ connect to the left semiedges of $\mathcal{B}_{i+1}$, and a dock index d_i, which specifies which left semiedge of $\mathcal{B}_i$ connects to the vertex u_i of $\mathcal{A}_i$.

Submultipoles and their compatible colorings. We now focus on normal 5-edge-colorings of a superposition $G_C(\mathcal{A}, \mathcal{B})$. To proceed, we first introduce the concepts of a submultipole and the restriction of a coloring to

a submultipole.
Let $M = (V, E, S)$ be a multipole. A multipole $M' = (V', E', S')$ is called a *submultipole* of M if $V' \subseteq V$, $E' \subseteq E$, and $S' \subseteq S \cup E_S$, where E_S represents the set of halves of edges in E. For a subset $V' \subseteq V$, a multipole $M' = M[V']$ is called an *induced* submultipole of M if:

- The vertex set is V'.
- The edge set E' includes all edges $e \in E$ where both endpoints belong to V'.
- The semiedge set S' includes all semiedges in M with endpoints in V', along with the halves of edges in E that have exactly one endpoint in V'.

Now, let σ be a normal 5-edge-coloring of a cubic multipole M. The *restriction* of σ to a submultipole M', denoted $\sigma' = \sigma|_{M'}$, is defined as follows:

- For each edge $e \in E'$, $\sigma'(e) = \sigma(e)$.
- For each semiedge $s \in S' \cap S$, $\sigma'(s) = \sigma(s)$.
- For each semiedge $s \in S' \setminus S$, $\sigma'(s) = \sigma(e_s)$, where e_s is the edge in M whose semiedge is s.

Finally, let $M_1, \dots, M_k$ be cubic submultipoles of a cubic multipole M, and let σ_i be a normal 5-edge-coloring of M_i for $i = 1, \dots, k$. Let M' denote the submultipole of M induced by the union of vertices $\cup_{i=1}^{k} V(M_i)$. The colorings σ_i are said to be *compatible* if there exists a normal 5-edge-coloring σ' of M' such that $\sigma'|_{M_i} = \sigma_i$ for every $i = 1, \dots, k$.

Our approach to the coloring of a superposition Throughout the paper, we assume that a snark G has a normal 5-edge-coloring σ, and we wish to extend this coloring to a superposition $\tilde{G} \in \mathcal{G}_C(\mathcal{A}, \mathcal{B})$. To be more precise, let C be a cycle in G and denote by M_{int} a submultipole of G induced by $V(G) \backslash V(C)$. By $\tilde{\sigma}_{\text{int}}$ we denote the restriction of σ to the submultipole M_{int}. Notice that M_{int} is a submultipole of the superposition $\tilde{G}$ also. We aim to construct a normal 5-edge coloring $\tilde{\sigma}$ of a superposition $\tilde{G}$ such that the restriction of $\tilde{\sigma}$ to M_{int} equals $\tilde{\sigma}_{\text{int}}$. We achieve this by constructing a normal 5-edge-coloring $\tilde{\sigma}_i$ of each superedge $\mathcal{B}_i$ with particular properties which assure that $\tilde{\sigma}_{i-1}$, $\tilde{\sigma}_i$ and $\tilde{\sigma}_{\text{int}}$ are compatible in $\tilde{G}$ for every i. This will directly imply the compatibility of all $\tilde{\sigma}_i$ and $\tilde{\sigma}_{\text{int}}$, i.e. the existance of a normal 5-edge-coloring of $\tilde{G}$.

3 Superposition by the Petersen graph

The Petersen graph, being the smallest snark, serves as a natural starting point for our study of superpositions where superedges are derived from this graph. Specifically, in this section, we consider the superpositions defined in Definition 2.1, where $H = P_{10}$ for each edge $e_i \in E(C)$, i.e., $\mathcal{B}_i = (P_{10})_{u,v}$ for all $i = 0, \dots, g-1$.

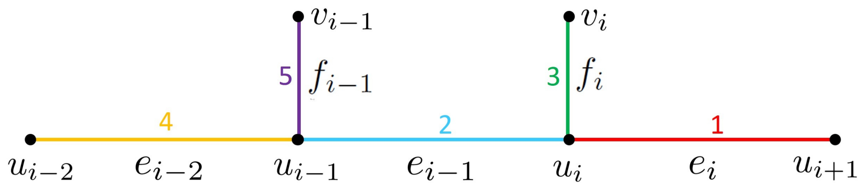

Figure 4: A normal 5-edge-coloring σ of the edges in G incident to the vertices u_{i-1} and u_i.

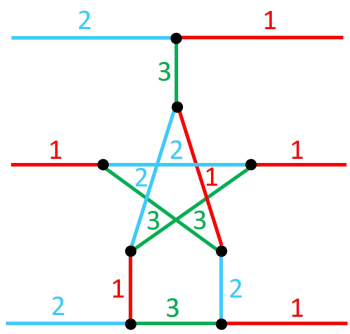

Figure 5: A normal 5-edge-coloring of $\mathcal{B}_i$ that is both right-side σ-monochromatic and left-side σ-compatible, assuming $d_i = 2$.

To proceed, we first define the concept of the color scheme of a semiedge. For a semiedge s in a cubic multipole M, the *color scheme* $\sigma[s]$ is given by $\sigma[s] = (i, \{j, k\})$, where i is the color of s and $\{j, k\}$ is the set of the two colors on the (semi)edges adjacent to s. Now, let s' be another semiedge in a multipole with the color scheme $\sigma[s'] = (i', \{j', k'\})$. The color schemes $\sigma[s]$ and $\sigma[s']$ are said to be *consistent*, denoted by $\sigma[s] \approx \sigma[s']$, if $i' = i$ and either $\{j', k'\} = \{j, k\}$ or $\{j', k'\} \cap \{i, j, k\} = \emptyset$. If the color schemes $\sigma[s]$ and $\sigma[s']$ are consistent, then identifying the semiedges s and s' when ”gluing” two multipoles results in either a poor or rich edge.

Next, we define a specific type of coloring for superedges used in our construction. A normal 5-edge-coloring $\tilde{\sigma}_i$ of $\mathcal{B}_i$ is called *right-side σ-monochromatic* if for every $j \in \{1,2,3\}$, the condition $\tilde{\sigma}_i[s^r_j] \approx (\sigma(e_i), \{\sigma(e_{i-1}), \sigma(f_i)\})$ holds. For instance, if the edge-coloring σ of G around vertex u_i is as shown in Figure 4, then the coloring of $\mathcal{B}_i$ depicted in Figure 5 is an example of a right-side σ-monochromatic coloring of $\mathcal{B}_i$. A normal 5-edge-coloring $\tilde{\sigma}_i$ of $\mathcal{B}_i$ is said to be *left-side σ-compatible* if the following conditions are satisfied:

- $\tilde{\sigma}_i[s^l_{d_i}] \approx (\sigma(e_i), \{\sigma(e_{i-1}), \sigma(f_i)\})$,
- $\tilde{\sigma}_i[s^l_j] \approx (\sigma(e_{i-1}), \{\sigma(e_i), \sigma(f_i)\})$ for every $j \in \{1,2,3\} \setminus \{d_i\}$, and
- there exists a Kempe $(\sigma(e_{i-1}), \sigma(f_i))$-chain P^l that connects the two left semiedges s^l_j where $j \neq d_i$.

To illustrate this concept, assume the edge-coloring σ of G is as shown in Figure 4. If $d_i = 2$, then the coloring of $\mathcal{B}_i$ depicted in Figure 5 is an example of a left-side σ-compatible coloring of $\mathcal{B}_i$.

Remark 3.1. Let $\tilde{\sigma}_{i-1}$ be a right-side σ-monochromatic coloring of $\mathcal{B}_{i-1}$, $\tilde{\sigma}_i$ a left-side σ-compatible coloring of $\mathcal{B}_i$, and $\tilde{\sigma}_{\text{int}}$ the restriction of σ to M_{int}. The following holds:

- If $\mathcal{A} = A$, then $\tilde{\sigma}_{i-1}$, $\tilde{\sigma}_i$, and $\tilde{\sigma}_{\text{int}}$ are compatible, meaning they combine seamlessly.
- If $\mathcal{A} = A'$, compatibility still holds, but $\tilde{\sigma}_i$ must be replaced by a modified coloring $\tilde{\sigma}'_i$, which is obtained by swapping colors along the Kempe chain P^l.

Therefore, if every superedge $\mathcal{B}_i$ admits a normal 5-edge-coloring that is simultaneously right-side σ-monochromatic and left-side σ-compatible, a normal 5-edge-coloring of the entire superposition can be achieved. However, when $d_i = 1$, such a coloring of $\mathcal{B}_i$ would result in the semiedges s^l_1 and s^r_1 having the same color. Since these semiedges are incident to the same vertex, this would violate the condition of a proper coloring. Thus, we conclude the following.

Observation 3.2. *A superedge $\mathcal{B}_i$ cannot admit a normal 5-edge-coloring that is simultaneously right-side σ-monochromatic and left-side σ-compatible in the case where $d_i = 1$.*

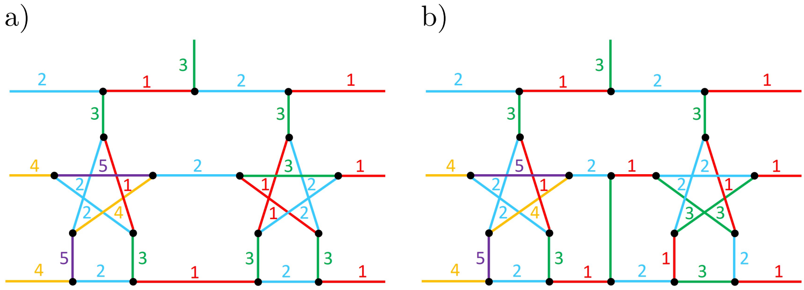

Figure 6: Given a coloring σ of G as shown in Figure 4, this figure illustrates σ-compatible colorings: $\tilde{\sigma}_{i-1}$, which is left-side σ-compatible, and $\tilde{\sigma}_i$, which is right-side σ-monochromatic. The two cases shown are: a) $\mathcal{A}_i = A$ and b) $\mathcal{A}_i = A'$, both with $d_{i-1} = d_i = 1$.

To address this issue, we group certain pairs of consecutive superedges into larger "chunks." By doing so, it becomes possible to ensure that these larger chunks are both right-side σ-monochromatic and left-side σ-compatible. This concept is illustrated in Figures 6 and 7. Using this strategy, we can demonstrate that when $d_i \neq 1$ for at least one superedge $\mathcal{B}_i$, the superedges of the superposition $G_C(\mathcal{A}, \mathcal{B})$ can be partitioned into a combination of single superedges and pairs of consecutive superedges, such that:

- A single superedge is assigned a normal 5-edge-coloring that is both right-side σ-monochromatic and left-side σ-compatible, as shown in Figure 5.
- A pair of consecutive superedges forms a larger chunk that is colored to be right-side σ-monochromatic and left-side σ-compatible, as depicted in Figures 6 and 7.

With additional refinements, this approach leads to the following result:

Theorem 3.3. *[26] Let G be a snark, σ a normal 5-edge-coloring of G, C a cycle of length g in G, and $\tilde{G} \in \mathcal{G}_C(\mathcal{A}, \mathcal{B})$ a superposition of G. If there exists at least one $i \in \{0, \ldots, g-1\}$ such that $p_i(1) \neq 1$ or $d_i \neq 1$, then $\tilde{G}$ admits a normal 5-edge-coloring $\tilde{\sigma}$ with at least 18 poor edges.*

On the other hand, if $d_i = 1$ for every superedge $\mathcal{B}_i$, the above approach succeeds only when C is an even-length cycle. This limitation arises because an odd number of superedges cannot be grouped into pairs for coloring as shown in Figure 6, and a single superedge $\mathcal{B}_i$ with $d_i = 1$ cannot have

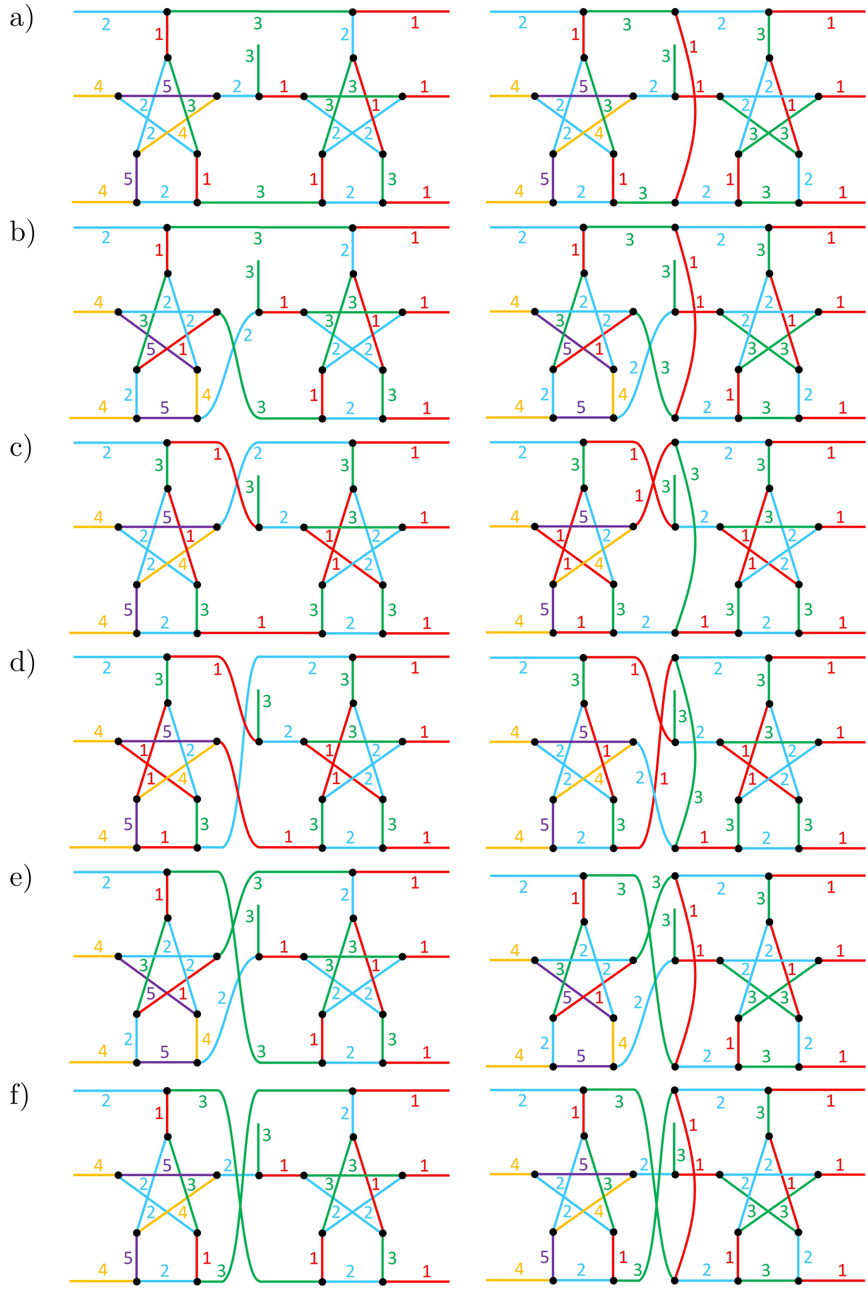

Figure 7: For the normal 5-edge-coloring σ of G shown in Figure 4, this figure illustrates normal 5-edge-colorings of $\mathcal{B}_{i-1}$ and $\mathcal{B}_i$ that are compatible with σ. The left column corresponds to $\mathcal{A}_i = A$, and the right column corresponds to $\mathcal{A}_i = A'$. These configurations are shown for $d_i = 2$ and the following permutations p_{i-1}: a) $(1, 2, 3)$, b) $(1, 3, 2)$, c) $(2, 1, 3)$, d) $(2, 3, 1)$, e) $(3, 1, 2)$, and f) $(3, 2, 1)$.

a normal 5-edge-coloring that is both right-side σ-monochromatic and left-side σ-compatible, as stated in Observation 3.2. Consequently, in such cases, the following theorem provides the best result achievable by our approach:

Theorem 3.4. *[24] Let G be a snark, σ a normal 5-edge-coloring of G, C a cycle of length g in G, and $\tilde{G} \in \mathcal{G}_C(\mathcal{A}, \mathcal{B})$ a superposition of G. If $p_i = (1,2,3)$ and $d_i = 1$ for every $i \in \{0, \ldots, g-1\}$, then for even g, there exists a normal 5-edge-coloring $\tilde{\sigma}$ of $\tilde{G}$ with at least 18 poor edges.*

Next, we consider the case of a superposition along an odd-length cycle C in a snark G, where $p_i = (1,2,3)$ and $d_i = 1$ for every superedge $\mathcal{B}_i$. The question arises whether it is generally impossible to extend a normal 5-edge-coloring of G to such a superposition $\tilde{G}$, or if this limitation is specific to our approach. To explore this, we analyze the Petersen graph $G = P_{10}$.
The Petersen graph G, being vertex-transitive, has (up to isomorphism) a single normal 5-edge-coloring, in which every edge is rich. It can be verified computationally that the following holds:

Observation 3.5. *[24] Let G be the Petersen graph and C a cycle of length 5 in G. Consider the superposition $G_C(\mathcal{A}, \mathcal{B})$ of G such that $\mathcal{A}_i = A$ and $\mathcal{B}_i = (P_{10})_{u,v}$ with $p_i = (1,2,3)$ and $d_i = 1$ for all $i \in \{0, \ldots, 4\}$. It is not possible to extend the normal 5-edge-coloring of G to the superposition $G_C(\mathcal{A}, \mathcal{B})$ without altering the colors of edges in G outside C.*

It is important to note that the above observation does not contradict the Petersen Coloring Conjecture. The superposition $G_C(\mathcal{A}, \mathcal{B})$ does admit a normal 5-edge-coloring; however, this coloring cannot be obtained as an extension of the normal 5-edge-coloring of $G = P_{10}$. In other words, while the superposition can be colored normally, doing so requires changing the colors of edges in G outside the cycle C.

4 Superposition by any snark

In the approach discussed in the previous section, a problem arises when the Petersen graph P_{10} is used as a superedge due to its diameter being two. This implies that any two vertices in P_{10} are at a distance of at most 2. Consequently, for at least one choice of the dock d_i, an edge of the snark G belonging to the cycle C is replaced in the superposition by a path of length two. It is evident that such a path cannot be assigned the same color as the corresponding edge in C without violating the properness of the coloring.
To address this issue, we use larger snarks as superedges and restrict attention to pairs of vertices u, v in these snarks such that $d(u, v) \geq 3$. In other

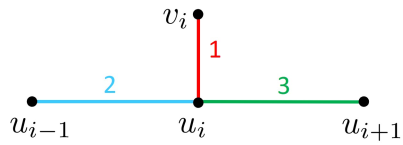

Figure 8: A coloring σ of the edges incident to a vertex u_i of the cycle C in G. For this coloring σ, all colorings of $\mathcal{B}_i$ consistent with the color schemes shown in Figure 9 are σ-compatible.

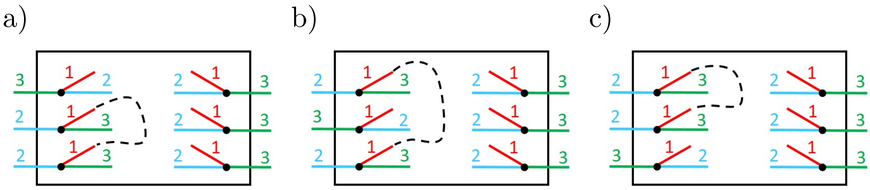

Figure 9: The color scheme κ_j for: a) $j = 1$, b) $j = 2$, c) $j = 3$. The dashed curve represents the Kempe chain P^l.

words, we focus on superedges of the form $H_{u,v}$, where H is a snark with a diameter of at least three, and u, v are vertices in H satisfying $d(u, v) \geq 3$. To generalize the approach from the previous section for any snark used as a superedge, we extend the notion of (consistent) color schemes from semiedges to connectors and superedges. Let $\tilde{G} \in \mathcal{G}_C(\mathcal{A}, \mathcal{B})$ be a superposition of G, $\mathcal{B}_i$ a superedge of $\tilde{G}$, and $\tilde{\sigma}_i$ a 5-edge-coloring of $\mathcal{B}_i$. The *color scheme* of the left and right connector of $\mathcal{B}_i$ is defined as follows:

$$\tilde{\sigma}_i[S^l] = (\tilde{\sigma}_i[s^l_1], \tilde{\sigma}_i[s^l_2], \tilde{\sigma}_i[s^l_3]) \quad \text{and} \quad \tilde{\sigma}_i[S^r] = (\tilde{\sigma}_i[s^r_1], \tilde{\sigma}_i[s^r_2], \tilde{\sigma}_i[s^r_3]),$$

respectively. The *color scheme* of a superedge $\mathcal{B}_i$ is then defined as:

$$\tilde{\sigma}_i[\mathcal{B}_i] = (\tilde{\sigma}_i[S^l], \tilde{\sigma}_i[S^r]),$$

and is illustrated in Figure 8.
Let $\mathcal{B}_i$ be a superedge of $G_C(\mathcal{A}, \mathcal{B})$, and let $\tilde{\sigma}_i$ and $\tilde{\sigma}'_i$ be two normal 5-edge-colorings of $\mathcal{B}_i$. The colorings $\tilde{\sigma}_i$ and $\tilde{\sigma}'_i$ are *consistent* on the left connector S^l of $\mathcal{B}_i$, denoted by $\tilde{\sigma}_i[S^l] \approx \tilde{\sigma}'_i[S^l]$, if:

$$\tilde{\sigma}_i[s^l_j] \approx \tilde{\sigma}'_i[s^l_j] \quad \text{for all } j = 1, 2, 3.$$

The consistency of $\tilde{\sigma}_i$ and $\tilde{\sigma}'_i$ on the right connector S^r of $\mathcal{B}_i$ is defined analogously. Finally, the colorings $\tilde{\sigma}_i$ and $\tilde{\sigma}'_i$ are said to be *consistent* on $\mathcal{B}_i$, denoted by $\tilde{\sigma}_i[\mathcal{B}_i] \approx \tilde{\sigma}'_i[\mathcal{B}_i]$, if they are consistent on both the left and right connectors. When this condition holds, we also say that the color schemes $\tilde{\sigma}_i[\mathcal{B}_i]$ and $\tilde{\sigma}'_i[\mathcal{B}_i]$ are consistent.

4.1 Right colorings

A normal 5-edge-coloring of $\mathcal{B}_i$ is called *j-right* if it is consistent with the color scheme κ_j from Figure 9 and there exists a Kempe $(2,1)$-chain P^l connecting the pair of left semiedges distinct from s^l_j. A superedge $\mathcal{B}_i$ is classified as follows:

- *Dock-right*: If it is j-right for $j = d_i$.
- *Doubly-right*: If it is j-right for at least two distinct values of j.
- *Fully-right*: If it is j-right for all $j \in \{1,2,3\}$.

A j-right coloring of $\mathcal{B}_i$ is σ-compatible with the coloring σ of the cycle C in G, as shown in Figure 8, provided that the dock of $\mathcal{B}_i$ is $d_i = j$. For any other coloring σ of G, a σ-compatible coloring of $\mathcal{B}_i$ can be derived from a j-right coloring by applying a color permutation and/or swapping colors along P^l. Additionally, a j-right coloring of $\mathcal{B}_i$ is compatible with j-right colorings of $\mathcal{B}_{i-1}$ and $\mathcal{B}_{i+1}$, provided all these colorings are σ-compatible. Based on this, we establish the following theorem:

Theorem 4.1. *[25] Let G be a snark with a normal 5-edge-coloring σ, C a cycle of length g in G, and $\tilde{G} \in \mathcal{G}_C(\mathcal{A},\mathcal{B})$ a superposition of G. If every superedge $\mathcal{B}_i$ is dock-right for $i = 0,\dots,g-1$, then $\tilde{G}$ admits a normal 5-edge-coloring.*

As an immediate consequence of Theorem 4.1, we obtain the following corollary:

Corollary 4.2. *[25] Let G be a snark with a normal 5-edge-coloring σ, C a cycle of length g in G, and $\tilde{G} \in \mathcal{G}_C(\mathcal{A},\mathcal{B})$ a superposition of G. If every superedge $\mathcal{B}_i$ is fully-right for $i = 0,\dots,g-1$, then $\tilde{G}$ admits a normal 5-edge-coloring.*

The sufficient conditions outlined in Theorem 4.1 and Corollary 4.2 hold for any snark H used as a superedge and for any pair of vertices u, v in H such that $d(u,v) \geq 3$. To demonstrate the broad applicability of these conditions, in the next subsection we describe a large family of snarks H for which these conditions are satisfied.
Before proceeding, we note that the extension of a coloring σ of a snark G to a superposition $\tilde{G}$ can be performed independently along multiple vertex-disjoint cycles. Recall that an *even subgraph* of a graph G is a subgraph

where every vertex has an even degree. Based on this, we state the following formal remark:

Remark 4.3. Theorem 4.1 remains valid when C is an even subgraph of G.

4.2 Right colorings and hypohamiltonian snarks

We now demonstrate that the condition in Theorem 4.1 holds for hypohamiltonian snarks. A graph H is said to be *hypohamiltonian* if H itself is not hamiltonian, but the removal of any vertex $v \in V(H)$ results in a graph $H - v$ that is hamiltonian. It is known that an infinite family of snarks, the so-called Flawer snarks, are hypohamiltonian [6]. Furthermore, in [17] hypohamiltonian snarks with cyclic connectivity 5 and 6 are constructed for all but finitely many even orders. Thus, there exist infinitely many snarks for which the following proposition holds.

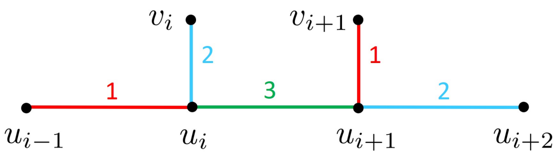

Figure 10: A coloring σ of the edges incident to vertices u_i and u_{i+1} of a cycle C in G. All j, k-left colorings of $\mathcal{B}_i$ consistent with the color schemes in Figure 11 are σ-compatible with this coloring σ of G.

Proposition 4.4. *[25] Let H be a hypohamiltonian snark, and let x, y be a pair of non-adjacent vertices in H. Then, $H_{x,y}$ is j-right for at least one $j \in \{1, 2, 3\}$.*

We outline the proof of this result. For any hamiltonian cycle in $H - y$, the cycle must be of odd length. Thus, its edges can be alternately colored by 1 and 2, except for the two edges incident to x, which are both colored by 2. All other edges in H are assigned the color 3. Removing the vertices x and y from H to obtain $H_{x,y}$, and preserving the colors of (semi)edges in $H_{x,y}$ as in H, results in a j-right coloring of $H_{x,y}$ for some $j \in \{1, 2, 3\}$. Note, however, that the specific value of j is not determined.
As a consequence, Theorem 4.1 and Proposition 4.4 together imply that a superposition admits a normal 5-edge-coloring for any hypohamiltonian snark H used as a superedge, and for any pair of vertices $x, y \in H$ such that $d(x, y) \geq 3$. However, this result applies only to certain ways of identifying

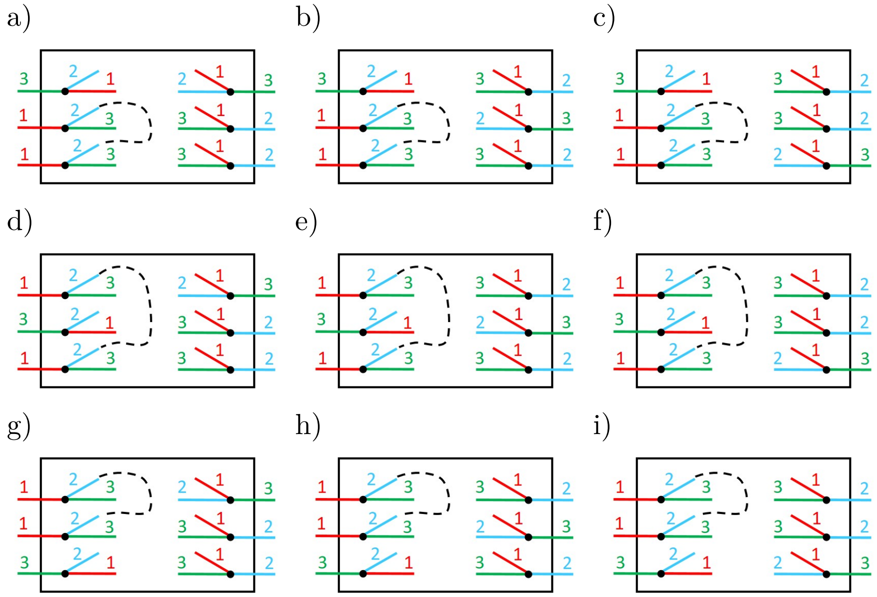

Figure 11: The color scheme $\tau_{j,k}$ for: a) $(j,k) = (1,1)$, b) $(j,k) = (1,2)$, c) $(j,k) = (1,3)$, d) $(j,k) = (2,1)$, e) $(j,k) = (2,2)$, f) $(j,k) = (2,3)$, g) $(j,k) = (3,1)$, h) $(j,k) = (3,2)$, i) $(j,k) = (3,3)$.

semiedges. Moreover, it can be verified that for some choices of x, y already in H being the smallest Flower snark, the superedge $H_{x,y}$ is not fully-right. Hence, a more refined sufficient condition is needed.

4.3 Left colorings

A normal 5-edge-coloring of $\mathcal{B}_i$ is referred to as a *j,k-left coloring* if it is consistent with the color scheme $\tau_{j,k}$ shown in Figure 11 and there exists a Kempe $(1,2)$-chain P^l connecting a pair of left semiedges distinct from s^l_j. A superedge $\mathcal{B}_i$ is called *doubly-left* if, for every $j \in \{1,2,3\}$, it admits a j,k-left coloring for at least two distinct values of k. In other words, $\mathcal{B}_i$ is doubly-left if it has a normal 5-edge-coloring consistent with at least two color schemes from each row in Figure 11.
For a coloring σ of G as illustrated in Figure 10, a j,k-left coloring of $\mathcal{B}_i$ is σ-compatible, provided that $d_i = j$. If $\mathcal{A} = A$, such a coloring is also compatible with a right coloring of $\mathcal{B}_{i-1}$. If $\mathcal{A} = A'$, compatibility with a right coloring of $\mathcal{B}_{i-1}$ can be achieved by swapping colors along P^l. For other colorings σ of G, a σ-compatible coloring of $\mathcal{B}_i$ with similar properties can be obtained from a j,k-left coloring by applying a color permutation

and/or appropriate color swaps along P^l.
By leveraging the compatibility of left and right colorings, and the compatibility among right colorings, and partitioning the superedges into singletons or consecutive pairs, we establish the following result:

Theorem 4.5. *[25] Let G be a snark with a normal 5-edge-coloring σ, C a cycle of length g in G, and $\tilde{G} \in \mathcal{G}_C(\mathcal{A}, \mathcal{B})$ a superposition of G. If each superedge $\mathcal{B}_i$ is both doubly-right and doubly-left for every $i = 0, \ldots, g-1$, then $\tilde{G}$ admits a normal 5-edge-coloring.*

As with Theorem 4.1, Theorem 4.5 provides a sufficient condition for extending a normal 5-edge-coloring of G to its superposition. This condition applies to superedges of the form $H_{x,y}$, where H is any snark, and x, y are any vertices in H satisfying $d(x, y) \geq 3$. To illustrate the broad applicability of this condition, we present an infinite family of snarks to which it applies, namely, Flower snarks.
Furthermore, since the extension described in Theorem 4.5 can be applied independently along multiple vertex-disjoint cycles, it follows that Theorem 4.5 also holds when C is an even subgraph of G.

4.4 Left colorings and Flower snarks

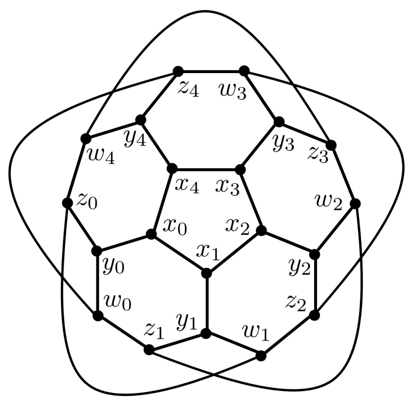

Figure 12: The Flower snark J_5.

A Flower snark J_r, for odd $r \geq 5$, is defined as a graph with the vertex set

$$V(J_r) = \bigcup_{i=0}^{r-1} \{x_i, y_i, z_i, w_i\},$$

and the edge set

$$E(J_r) = \bigcup_{i=0}^{r-1} \{x_i x_{i+1}, x_i y_i, y_i z_i, y_i w_i, z_i w_{i+1}, w_i z_{i+1}\},$$

where indices are taken modulo r. The Flower snark for $r = 5$ is depicted in Figure 12. We consider a superedge $H_{x,y}$, where $H = J_r$, and $x, y \in V(H)$ is any pair of vertices in H such that $d(x, y) \geq 3$.
To apply Theorem 4.5 to all superpositions using Flower snarks, we need to establish that the superedge $(J_r)_{x,y}$ is both doubly-right and doubly-left for every odd $r \geq 5$ and every pair of vertices x, y in J_r with $d(x, y) \geq 3$. A reduction method introduced by Hagglund and Steffen [7] allows us to limit our consideration to the Flower snark J_5 and specific pairs of vertices in J_7, as detailed in the following proposition, which is verified computationally (*in silico*).

Proposition 4.6. *[25] Let J_5 be the Flower snark, and let x, y be a pair of vertices in J_5 such that $d(x, y) \geq 3$. Then the superedge $(J_5)_{x,y}$ is both doubly-right and doubly-left. The same holds for a superedge $(J_7)_{x,y}$, where $(x, y) \in \{(x_0, x_3), (z_0, z_3), (z_3, z_0)\}$.*

Building on the above proposition and using the reduction method for larger Flower snarks, the application of Theorem 4.5 leads to the following result:

Theorem 4.7. *[25] Let G be a snark with a normal 5-edge-coloring σ, C an even subgraph of G, and $\tilde{G} \in \mathcal{G}_C(\mathcal{A}, \mathcal{B})$ a superposition of G. If $\mathcal{B}_i \in \{(J_r)_{x,y} : x, y \in V(J_r) \text{ and } d(x, y) \geq 3\}$ for every $e_i \in E(C)$, then $\tilde{G}$ admits a normal 5-edge-coloring.*

Since the Petersen Coloring Conjecture implies the Ford-Fulkerson Conjecture, Theorem 4.7 implies the results presented in [14].

5 Concluding remarks

The findings derived from our research on this topic have inspired us to propose a conjecture suggesting the validity of additional claims. To articulate this conjecture, we first introduce some essential definitions. A normal coloring of a cubic graph G is called a *strong coloring* if every edge in the graph is rich. The minimum number of colors needed to achieve a strong coloring of a cubic graph G is referred to as the *strong chromatic index* and is denoted by $\chi'_s(G)$. Let $\mathrm{NC}(G)$ represent the set of all normal 5-colorings of G. The Petersen Coloring Conjecture asserts that $\mathrm{NC}(G) \neq \emptyset$ for any bridgeless cubic graph G.

Assuming the Petersen Coloring Conjecture is true, meaning that $\mathrm{NC}(G)$ is non-empty for every bridgeless cubic graph G, we define $\mathrm{poor}(G)$ as the highest number of poor edges found across all colorings in $\mathrm{NC}(G)$. In examining $\mathrm{poor}(G)$, it is particularly insightful to first analyze the 3-cycles and 4-cycles present in a snark G, provided such cycles exist within G.

Remark 5.1. [24] Let G be a bridgeless cubic graph and σ a normal 5-coloring of G. If G contains a 3-cycle C, then every edge in C is poor in σ. Similarly, if G contains a 4-cycle C, then either 2 or 4 edges of C must be poor in σ.

It follows that any graph admitting a normal 5-coloring devoid of poor edges must have a girth of at least 5. Denote by P_{10}^{Δ} the graph obtained from P_{10} by truncating one of its vertices. A normal 5-edge-coloring of P_{10}^{Δ} has at least 3 poor edges, and a straightforward verification confirms that it has exactly 3 poor edges. This observation motivates the following conjecture.

Conjecture 5.2. [24] Let G be a bridgeless cubic graph. If $G \neq P_{10}$, then $\mathrm{poor}(G) > 0$. Moreover, if $G \neq P_{10}, P_{10}^{\Delta}$, then $\mathrm{poor}(G) \geq 6$.

Regarding the two sufficient conditions for superpositions by any snark H and any pair of vertices x, y in H with $d(x, y) \geq 3$, the sufficient condition in Theorem 4.1 is weaker than that in Theorem 4.5. Nonetheless, it is applicable to superpositions by infinitely many distinct snarks, specifically to all hypohamiltonian snarks used as superedges, although not to all possible ways of semiedge identification. For instance, since Flower snarks are hypohamiltonian, Theorem 4.1 implies that many snarks superposed by Flower snarks admit a normal 5-edge-coloring. In contrast, the condition of Theorem 4.5 is more stringent. When applied to snarks superposed by Flower snarks, it guarantees that all such superpositions have a normal 5-edge-coloring.

For Flower snark superedges, the sufficient condition in Theorem 4.5 can be reformulated to involve the corresponding 2-factors, whose existence is then verified computationally. A promising avenue for future research is to establish that all snarks, or certain broad families of snarks, possess the required 2-factorizations.

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Intrinsic geometry of cyclic polygons via new Brahmagupta's formula revisited

Dragutin Svrtan

Abstract

In this paper, we give full details for an intrinsic approach, using the author's New Brahmagupta formulas, to the computation of Heron polynomials for cyclic polygons (up to $n = 8$). A less complete account was already given in [20] (and used by S. Moritsugu, see ref. [27]) following the author's talk at the International Congress of Mathematicians in Hyderabad, India, in 2010. We also mention a new approach by multivariate discriminants based on the fact that the cyclic polygons are critical points of the area functional.

1 Introduction

Finding explicit equations for the area or circumradius of polygons inscribed in a circle in terms of side lengths is a classical subject (cf. [1]). For triangle / cyclic quadrilaterals, we have the famous Heron / Brahmagupta formulae. In 1994. D. P. Robbins found a minimal area equation for cyclic pentagons/hexagons by a method of undetermined coefficients (cf. [5]). This method could hardly be used for heptagons due to computational complexity (143307 equations).

In [8], by using covariants of binary quintics, a concise minimal heptagon/octagon area equation was obtained as a quotient of two resultants, which in expanded form has almost one million terms. It is not clear if this approach could be effectively used for cyclic polygons with nine or more sides.

In [15] and [28], by using the Wiener-Hopf factorization approach, we have obtained a very explicit minimal heptagon/octagon circumradius equation

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(only 13 pages formula) in Pellian form (= a combination of two squares of smaller polynomials whose coefficients have at most four digits). A nonminimal area equation is also obtainable by this method. Both methods are somehow external.

But, based on our new intermediate Brahmagupta formulas (2.6) and (2.7), we have succeeded in finding a direct intrinsic proof of the Robbins formulas for the area (and also for circumradius and area times circumradius) of cyclic hexagons.

Earlier, an intricate direct elimination of diagonals for cyclic hexagons was painful (see the footnote on the page 117) (the case of a pentagon was much easier, cf. [21]).

We also get a simple(st) system of equations ($EQ1$, $EQ2$, $EQ3$ on page 121) for the area (and area times circumradius) of cyclic octagons.

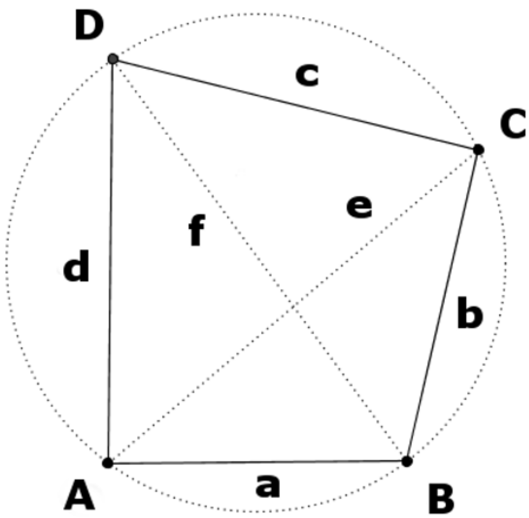
It seems remarkable that our approach, with the help of Gröbner basis techniques, leads to minimal equations (for any concrete instances we have tested), which is not the case with the iterated resultants approach.

Inspired by our observation on page 119 at the end we present a new method of multivariate discriminants, for finding area equation for cyclic octagons, of a master equation by using the result (cf. [16]) that cyclic polygons are critical points of the area functional.

For reader convenience we recommend a somewhat older survey [9] by I. Pak and references [22]– [27] by S. Moritsugu who used our reference [20] in [27].

We hope that our method of dissecting cyclic polygons into cyclic quadrilaterals is concordant with well known Grothendieck's well-known reconstruction principle.

2 Cyclic quadrilaterals



We first recall some basic formulas for cyclic quadrilaterals $ABCD$ with sides and diagonals of lengths $a = |AB|$, $b = |BC|$, $c = |CD|$, $d = |DA|$ and $e = |AC|$, $f = |BD|$ whose vertices lie on a circle of radius R.

- **Ptolemy's relation** (convex case):

$$ef = ac + bd \tag{2.1}$$

- **Dual Ptolemy's relation**:

$$(ab + cd)e = (ad + bc)f \quad (= 4SR) \tag{2.1'}$$

- **Diagonal equation**:

$$(ab + cd)\mathbf{e^2} = (ac + bd)(ad + bc) \tag{2.2}$$

- **Area equation** (Brahmagupta's formula, 625. AD):

$$16S^2 = 2(a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2)-a^4-b^4-c^4-d^4+8abcd \tag{2.3}$$

which, in a more popular form, reads as

$$16S^2 = (-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d) \tag{2.3'}$$

- **Circumradius equation** (Parameshavara's formula, 1400. AD):

$$R^2 = \frac{(ab+cd)(ac+bd)(ad+bc)}{(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d)} \tag{2.4}$$

- **Area times circumradius equation**:
  Let $Z = 4SR$, then

$$Z^2 = (ab + cd)(ac + bd)(ad + bc) \tag{2.5}$$

  which in a case of a triangle ($d = 0$) reduces to the well known relation

$$4SR = abc. \tag{2.5'}$$

For the reader's convenience, one of the simplest methods for obtaining Brahmagupta's formula makes use of trigonometry: for the interior angles at B and D we have $B + D = 180°$, implying $\cos D = -\cos B$, $\sin D = \sin B$. By the Law of Cosine, we obtain $2(ab + cd)\cos B = a^2 + b^2 - c^2 - d^2$. For area $S = \frac{1}{2}ab\sin B + \frac{1}{2}cd\sin D = \frac{1}{2}(ab + cd)\sin B$. Hence $16S^2 = (2ab+2cd)^2-(a^2+b^2-c^2-d^2)^2 = (2s-2a)(2s-2b)(2s-2c)(2s-2d)$, where $2s = a+b+c+d$. This completes the classical proof of Brahmagupta's formula.

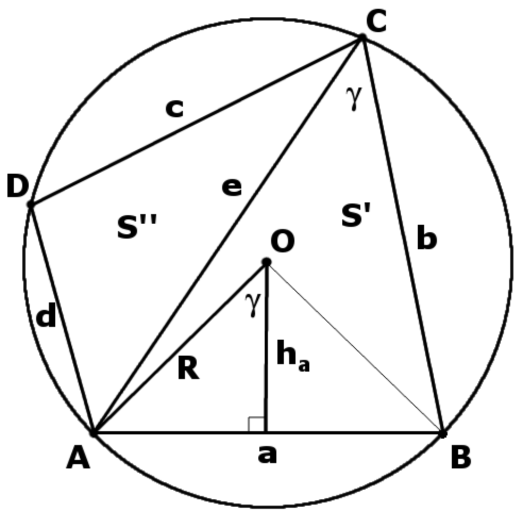
Our main contribution is the following discovery: the Key Lemma and a new (atomic) Brahmagupta's formula.
This lemma will be crucial in all our subsequent calculations concerning the elimination of diagonals in cyclic polygons.

Key Lemma: (Intermediate Brahmagupta's formula)
In any convex cyclic quadrilateral, we have

$$\boxed{8Sh_a = 2bcd + (b^2 + c^2 + d^2 - a^2)a} \tag{2.6}$$

where h_a denotes the height (positive or negative) of the center of the circumcircle with respect to the side AB.

In the case of a nonconvex quadrilaterals, we can formally obtain all the relations by simply allowing side lengths to be negative (e.g. by replacing a with $-a$).



Proof of the Key Lemma. $S = S' + S'' \Rightarrow$
$4RS = 4RS' + 4RS'' = abe + cde$ (Dual Ptolemy's relation)

By Law of Cosine, dual Ptolemy's and Diagonal equation for $h_a \geq 0$ we have: $h_a = R\cos\gamma = R\dfrac{b^2 + e^2 - a^2}{2be} = \dfrac{(ab + cd)(b^2 + e^2 - a^2)}{8Sb}$

$$= \frac{2bcd + (b^2 + c^2 + d^2)a - a^3}{8S}.$$

(Case $h_a < 0$ is similar.) ■
Let $S_a = \frac{ah_a}{2}$ be the signed area of the characteristic triangle $\triangle OAB$ determined by the side AB (of length a) and circumcenter O of a cyclic quadrilateral $ABCD$. Then we get

Corollary 2.1. *(New Brahmagupta's formulas)*

$$\boxed{16SS_a = a^2(b^2 + c^2 + d^2 - a^2) + 2abcd} \tag{2.7}$$

and three more formulas, by cyclically permuting a, b, c and d.

Note that by adding all four such formulas we get the original Brahmagupta's formula because

$$S = S_a + S_b + S_c + S_d.$$

For general quadrilaterals in a plane, we have:

- **Bretschneider's formula ([2]) or Staudt's formula (1842)**:

$$16S^2 = 4e^2f^2 - (a^2 - b^2 + c^2 - d^2)^2. \tag{2.8}$$

For cyclic quadrilaterals, in view of (2.1), it gives another form of (2.3):

$$16S^2 = 4(ac+bd)^2 - (a^2 - b^2 + c^2 - d^2)^2. \tag{2.3''}$$

The formula (2.8) is the simplest formula for the area of the quadrilateral in terms of its sides and diagonals. But there are infinitely many other ways to do so, since these 6 quantities satisfy Euler's four-point relation

$$\begin{aligned} &e^2 f^2 (a^2 + b^2 + c^2 + d^2 - e^2 - f^2) = \\ &= e^2(a^2 - b^2)(d^2 - c^2) + f^2(a^2 - d^2)(b^2 - c^2) + \\ &\quad + (a^2 - b^2 + c^2 - d^2)(a^2c^2 - b^2d^2) \end{aligned} \tag{2.9}$$

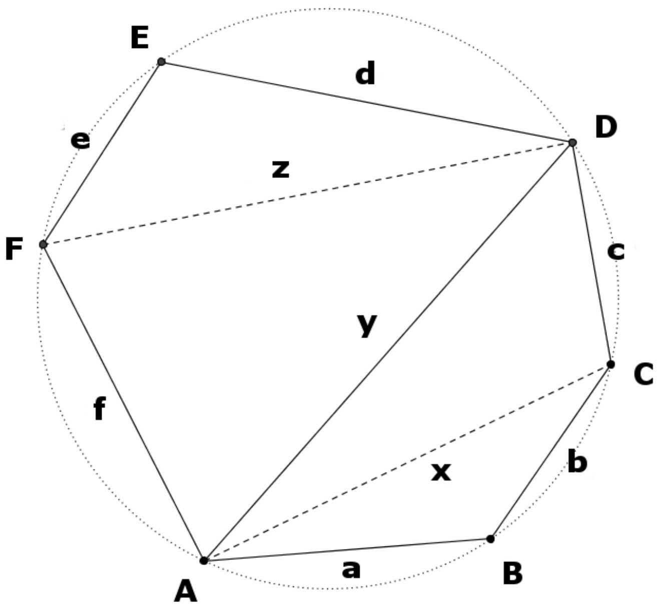
This is only a quadratic equation with respect to a square of each parameter. The Euler's four point relation follows from the Cayley–Menger determinant for the volume V of a tetrahedron with edges of lengths a, b, c, d, e, f if we set $V = 0$.

Remark 2.2. In a solution of a problem by J.W.L.Glaisher: With four given straight lines to form a quadrilateral inscribable in a circle, A.Cayley (in 1874.) observed the following identity, equivalent to (2.9):

$$\begin{aligned} &\left[(a^2 + b^2 + c^2 + d^2 - e^2 - f^2)(ef + ac + bd) - 2(ad + bc)(ab + cd)\cdot\right] \\ &\cdot(ef - ac - bd) = \left[(ab + cd)e - (bc + ad)f\right]^2 \end{aligned} \tag{2.9'}$$

which directly shows that Ptolemy's relation (2.1) implies the dual Ptolemy's relation (2.1').

3 Cyclic hexagons



Cyclic hexagon $ABCDEF$ inscribed in a circle of radius R, with side lengths $a = |AB|$, $b = |BC|$, $c = |CD|$, $d = |DE|$, $e = |EF|$, $f = |FA|$, $\boldsymbol{y}$ = **main** diagonal, $\boldsymbol{x}$,$\boldsymbol{z}$ = **small** diagonals.

- **Main diagonal equation**

Let $y = |AD|$ denote the length of the main diagonal of the cyclic hexagon $ABCDEF$. Then we may think of the hexagon $ABCDEF$ as made up of two quadrilaterals with a common side AD, both having the same circumradius R. Thus using the formula (2.4) twice we get equality

$$\begin{aligned}&(R^2 =)\\ &\frac{(de+fy)(df+ey)(ef+dy)}{(-d+e+f+y)(d-e+f+y)(d+e-f+y)(d+e+f-y)} =\\ &= \frac{(ab+cy)(ac+by)(bc+ay)}{(-a+b+c+y)(a-b+c+y)(a+b-c+y)(a+b+c-y)}\end{aligned} \tag{3.1}$$

leading to a 7–th degree equation

$$\boxed{(def-abc)y^7+\cdots=0}$$

for the length of the main diagonal y.
With substitutions

$$u = a^2+b^2+c^2, \quad v = a^2b^2+a^2c^2+b^2c^2, \quad w = abc \tag{3.2}$$

$$U = d^2 + e^2 + f^2, \quad V = d^2e^2 + d^2f^2 + e^2f^2, \quad W = def \tag{3.2'}$$

we can express the area S' (resp. S'') of the quadrilateral $ABCD$ (resp. $ADEF$) as follows

$$16S'^2 = 4v - u^2 + 8wy + 2uy^2 - y^4, \;\; 16S''^2 = 4V - U^2 + 8Wy + 2Uy^2 - y^4 \tag{3.3}$$

Then (3.1) becomes equivalent to

$$\begin{aligned} P_6^{\text{main diag.}} \equiv -S''^2 \left(wy^3 + vy^2 + uwy + w^2\right) + \\ + S'^2 \left(Wy^3 + Vy^2 + UWy + W^2\right) = 0 \end{aligned} \tag{3.1'}$$

$$\text{(i.e.} \boxed{(w - W)y^7 + (v - V)y^6 + \cdots + (4v - u^2)W^2 - (4V - U^2)w^2 = 0)} \tag{3.1''}$$

By letting $f = 0$, we obtain the diagonal equation for a cyclic pentagon $ABCDE$:

$$\boxed{\begin{aligned} &P_5^{\text{diag.}} \equiv \\ &abc\ y^7 + (a^2b^2 + a^2c^2 + b^2c^2 - d^2e^2)y^6 + \cdots + a^2b^2c^2(d^2 - e^2) = 0 \end{aligned}}$$

(cf. Bowman [4]).

- **Small diagonal equation**

Let $x = |AC|$ denote the length of a "small" diagonal in the cyclic hexagon $ABCDEF$. By (2.2) we obtain the equation

$$(ab + cy)x^2 = (ac + by)(bc + ay)$$

by which we can eliminate y in our main diagonal equation (3.1"). This gives our small diagonal equation, which has degree 7 in x^2:

$$\begin{aligned} &P_6^{\text{small diag.}} \equiv \\ &(abc - def)(abd - cef)(abe - cdf)(abf - cde)x^{14} + (\ldots)x^{12} + \cdots + \\ &+(a^2 - b^2)^4(acd - bef)(ace - bdf)(acf - bde)(ade - bcf) \\ &(adf - bce)(aef - bcd) = 0 \end{aligned} \tag{3.4}$$

By letting $f = 0$ we obtain

$$P_6^{\text{small diag.}} \Big|_{f=0} = a^3b^3 P_5^{\text{diag.}} \left(P_5^{\text{diag.}}\right)^* \tag{3.4'}$$

where $P_5^{\text{diag.}} \equiv cde\ x^7 + \cdots = 0$ and $\left(P_5^{\text{diag.}}\right)^*$ is obtained by changing sign of an odd number of side lengths c, d, e.

- **Area equation**: Naive approach

A naive approach to get the area equation of cyclic hexagon would be to write the area S of our hexagon as

$$S = S' + S'' \tag{3.5}$$

Then by rationalizing the equation (3.5) we obtain an equation of degree 4 in y:

$$(S^2 + S'^2 - S''^2)^2 - 4S^2S'^2 = 0 \tag{3.6}$$

where S'^2 and S''^2 are given by Brahmagupta's formula (3.3). More explicitly, in terms of the squared area $A = (4S)^2$ we have

$$\begin{aligned} Q \equiv \\ &(A + 4(v - V) + U^2 - u^2 + 8(w - W)y + 2(u - U)y^2)^2 - \\ &- 4A(4v - u^2 + 8wy + 2uy^2 - y^4) = 0 \end{aligned} \tag{3.6'}$$

By computing the resultant of this equation and the main diagonal equation (3.1') w.r.t. y we obtain a degree 14 polynomial in A.

$$\text{Resultant}\left(Eq(3.6'),\, P_6^{\text{main diag.}},\, y\right) = F_1F_2$$

both of whose factors have degree 7 in A:

$$\begin{aligned} F_1 &= (w - W)^2A^7 + \cdots \\ F_2 &= A^7 + (7u^2 + 7U^2 - 10uU - 24v - 24V)A^6 + \cdots \end{aligned}$$

The true equation (obtained first by Robbins in 1994. by undetermined coefficients method) is given by F_2 (it has 2042 monomials), and the extraneous factor F_1 (which has 8930 monomials) is 4 time bigger[4].

- **Area equation**: new approach leading to an intrinsic proof.

The complications with the extraneous factor in the previous proof were probably caused by using squaring operation twice in order to get the equation (3.6) (or (3.6')). So we are searching a simpler equation relating the area S and the main diagonal. After a long struggle we obtained an extraordinary simple relationship given in the following

Key Lemma. The area S of the cyclic hexagon $ABCDEF$ and areas S' and S'' of the cyclic quadrilaterals $ABCD$ and $ADEF$ obtained by

[4]The computation with MAPLE 9.5 on a PC with 2GHz and 2GB RAM took $\approx$ 300 hours (in year 2004). Nowadays with MAPLE 12 on a 64–bit PC with 8GB it takes $\approx$ 3 hours.

subdivision with the main diagonal of length $y = |AD|$ satisfy the following relations:

a) $(y^3 - (a^2 + b^2 + c^2)y - 2abc)S'' + (y^3 - (d^2 + e^2 + f^2)y - 2def)S' = 0$

b) $(y^3 - (a^2 + b^2 + c^2)y - 2abc)S + ((a^2 + b^2 + c^2 - d^2 - e^2 - f^2)y + 2(abc - def))S' = 0$

Proof. a) Let $x = |AC|$, $y = |AD|$, $z = |DF|$. Let S'_1, S'_2 S''_1 and S''_2 be the areas of triangles ABC, ACD, ADF and DEF respectively. Then, by (2.5') we have $4S'_1R = abx$, $4S'_2R = cxy$, $4S''_1R = fyz$, $4S''_2R = dez$. So we have $4S'R = (ab + cy)x$, $4S''R = (fy + de)z$. This implies

$$\frac{S''}{S'} = \frac{fy + de}{ab + cy} \cdot \frac{z}{x}$$

The diagonal equation for the main diagonal $y = |AD|$ in the middle quadrilateral $ACDF$: $(cx + fz)y^2 = (cf + xz)(fx + cz)$ can be rewritten as

$$cx(y^2 - f^2 - z^2) = fz(-y^2 + c^2 + x^2)$$

Now we have

$$\frac{S''}{S'} = \frac{fy + de}{ab + cy} \cdot \frac{y^2 - f^2 - z^2}{x^2 + c^2 - y^2} \cdot \frac{c}{f} = \frac{c}{f} \frac{(fy + de)(y^2 - f^2) - (fy + de)z^2}{(ab + cy)(c^2 - y^2) + (ab + cy)x^2}$$

Finally we use the diagonal equations for small diagonals x and z in respective quadrilaterals

$$(ab + cy)x^2 = (ac + by)(bc + ay), \quad (fy + de)z^2 = (df + ey)(ef + dy)$$

and by simplifying we get

$$\frac{S''}{S'} = \frac{y^3 - (d^2 + e^2 + f^2)y - 2def}{2abc + (a^2 + b^2 + c^2)y - y^3}$$

b) follows from a) by substituting $S'' = S - S'$. $\square$

By writing the equation b) in Key Lemma with shorthand notations (3.2) and (3.2')

$$(y^3 - uy - 2w)S + ((u - U)y + 2(w - W))S' = 0$$

and multiplying it by $2S$, $2S'$ respectively and using the relation

$$2SS' = S^2 + S'^2 - S''^2$$

obtained from (3.5) by squaring, we obtain the following

KEY EQUATIONS:

$$\boxed{Q_1 := 2(y^3 - uy - 2w)S^2 + ((u-U)y + 2(w-W))(S^2 + S'^2 - S''^2) = 0}$$

$$\boxed{Q_2 := (y^3 - uy - 2w)(S^2 + S'^2 - S''^2) + 2((u-U)y + 2(w-W))S'^2 = 0}$$

where S'^2 and S''^2 are given by Brahmagupta's formulas (3.3).

MAIN THEOREM. The resultant of the Key Equations with respect to y gives the minimal degree 7 equation for the squared area $A = (4S)^2$ of cyclic hexagon.

Proof. The minimal polynomial

$\alpha_6 = \text{Resultant}(Q_1, Q_2, y)/C = A^7 + (7(u^2+U^2) - 10uU - 24(v+V))A^6 + \cdots$

where $C = 4\left[4(W-w)^3 + (u-U)^3(wU - uW)\right]$.
Remark. Observe that $16Q_1 = [2A + 2(u-U)^2]y^3 + \cdots$
Similarly the polynomial Q in equation (3.6') has the form

$$Q = \left[4A + 2(u-U)^2\right] y^4 + \cdots$$

If we define

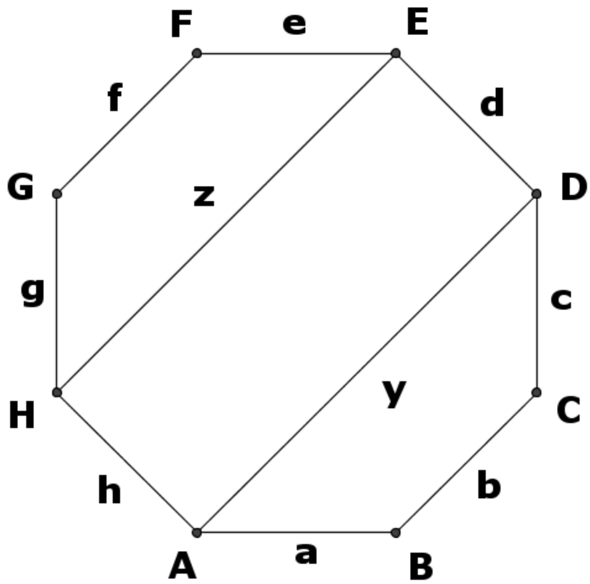
$$\begin{aligned} Q_3 &:= Q - 2\cdot 16Q_1 \\ &= 4(-uy^2 - 6wy - 4v + u^2)A + (4(v + w - V - W) + U^2 - u^2 + A)\cdot \\ &\quad \cdot (A + 4(v-V) + U^2 - u^2 + 8(w-W)y + 2(u-U)y^2) \end{aligned}$$

then we also get

$$\alpha_6 = \text{Resultant}(Q_3, 16Q_1, y)/(-8A^2)$$

New Observation! By using the fact that cyclic polygons are critical points of area (c.f. [16]) we can obtain New Theorem which uses the theory of discriminants $:= discrim(Q,y)/(2^14A^2) = A^7 + \cdots$. Where Q is given in (3.6').

4 Area equations of cyclic octagons and (heptagons)



We trisect cyclic octagon $ABCDEFGH$, by two diagonals AD and EH into three quadrilaterals $ABCD$, $ADEH$ and $EFGH$ whose areas we denote by S_1, S_2 and S_3 respectively. The area S of $ABCDEFGH$ is then equal to

$$S = S_1 + S_2 + S_3 \tag{4.1}$$

By Key Lemma a) applied to hexagons $ABCDEH$ and $ADEFGH$ we obtain the following equations:

$$(2jz + (i + z^2)y - y^3)S_1 + (2w + uy - y^3)S_2 = 0 \tag{4.2}$$
$$(2jy + (i + y^2)z - z^3)S_3 + (2W + Uz - z^3)S_2 = 0 \tag{4.3}$$

where we have used the following abbreviations:

$$\begin{aligned} &y = |AD|, z = |EH| \\ &u = a^2 + b^2 + c^2, v = a^2b^2 + a^2c^2 + b^2c^2, w = abc \\ &U = e^2 + f^2 + g^2, V = e^2f^2 + e^2g^2 + f^2g^2, w = efg \\ &i = d^2 + h^2, j = dh \end{aligned} \tag{4.4}$$

Furthermore the Brahmagupta formulas for the 16 times squared areas $A_i = 16S_i^2$, $i = 1, 2, 3$ can be written now as follows:

$$A_1 = 4v - u^2 + 8wy + 2uy^2 - y^4 \tag{4.5}$$

$$A_2 = 4(j+yz)^2 - (y^2+z^2-i)^2 \tag{4.6}$$
$$A_3 = 4V - U^2 + 8Wz + 2Uz^2 - z^4 \tag{4.7}$$

(For A_1, A_3 cf. (3.3), for A_2 cf. (2.3”) from Preliminaries!)
By equating the circumradius formulas for cyclic quadrilaterals $ABCD$ and $ADEH$ (resp. $ABCD$ and $EFGH$) we obtain two equations:

$$\boxed{\begin{aligned}&EQ1 := (4v-u^2+8wy+2uy^2-y^4)(jzy^3+(iz^2+j^2)y^2+(i+z^2)jzy+\\&+(jz)^2)-(4(j+yz)^2-(y^2+z^2-i)^2)(wy^3+vy^2+uwy+w^2)=0\end{aligned}} \tag{4.8}$$

$$\boxed{\begin{aligned}EQ2 :=\ &(4v-u^2+8wy+2uy^2-y^4)(Wz^3+Vz^2+UWz+W^2)-\\&-(4V-U^2+8Wz+2Uz^2-z^4)(wy^3+vy^2+uwy+w^2)=0\end{aligned}} \tag{4.9}$$

Our next aim is to get one more equation (as simple as possible) relating the lengths y and z of diagonals and the squared area $A = 16S^2$ of our cyclic octagon. Here is a result of a many years long search:

Theorem 4.1. *(Fundamental equation involving area of cyclic octagons) Let $A = 16S^2$ be the squared area of any cyclic octagon. Then we have the following equation of degree 6 in y and z and linear in A:*

$$\boxed{EQ3 := \alpha\gamma(A+\eta) + 2(\alpha-\beta)(\delta-\gamma)A_2 = 0} \tag{4.10}$$

where

$$\begin{aligned}&\alpha = 2jz + iy + yz^2 - y^3, \quad \beta = 2w + uy - y^3\\&\gamma = 2jy + iz + y^2z - z^3, \quad \delta = 2W + Uz - z^3\\&\eta = u^2 + U^2 - i^2 - 4v - 4V + 4j^2 - 8wj - 8Wz + 8jyz+\\&\quad + 2(i-u)y^2 + 2(i-U)z^2 + 2y^2z^2\end{aligned}$$

Proof. We start by squaring the equation (4.1)

$$S^2 = S_1^2 + S_2^2 + S_3^2 + 2S_1S_2 + 2S_1S_3 + 2S_2S_3 \tag{4.11}$$

Solving (4.2) for S_1 and (4.3) for S_2 yields:

$$S_1 = -\frac{\beta}{\alpha}S_2, \quad S_3 = -\frac{\delta}{\gamma}S_2 \tag{4.12}$$

Then we substitute these only into the mixed terms of (4.11). This gives:

$$S^2 = S_1^2 + S_2^2 + S_3^2 + 2\left(-\frac{\beta}{\alpha} + \frac{\beta\delta}{\alpha\gamma} - \frac{\delta}{\gamma}\right)S_2^2$$

By multiplying the last equation by 16 and using that $A_i = 16S_i^2$, $A = 16S^2$ we obtain

$$\alpha\gamma(A - A_1 + A_2 - A_3) + 2(\alpha - \beta)(\delta - \gamma)A_2 = 0$$

and set

$$\eta = -A_1 + A_2 - A_3$$

and the result follows by (4.5), (4.6) and (4.7). $\square$

Remark 4.2. By using Gröbner basis for $\{EQ1, EQ2, EQ3\}$ we get minimal equation α_7 (α_8) for squared area ($A = 16\,Area^2$) of cyclic heptagons (octagons) in concrete instances very fast.

Remark 4.3. Maley M.F., Robins D.P. and Roskies J. ([8]) obtained explicit formulas for α_7 and α_8 in terms of elementary symmetric functions of sides lengths squared.

$$\alpha_7 = \frac{2^{101}5^5 Res(\widetilde{F}, \widetilde{G}, u_3)}{u_2^4 Res(\widetilde{F}_1, \widetilde{F}_2, u_3)}$$

Half a year later we have fully expanded α_7 which has 955641 terms with up to 40-digits coefficients (approx. 5000 pages).

Remark 4.4. For ζ_7, the $Z(= 4SR)$-polynomial, by a similar method, we obtained explicit formula with 31590 terms with up to 11 digits coefficients.

Remark 4.5. For $\rho_7 = R^2$-equation of cyclic heptagon, by a different technique, we obtained a 15 pages output in a condensed (Pellian) form – a quadratic form of two smaller polynomials whose coefficients have up to 4 digits coefficients in terms of new quantities (which are certain linear combinations of elementary symmetric functions of side lengths squared) published explicitly in [28].

5 Area equations for cyclic octagons by using bivariate discriminants

We start with a cyclic octagon $ABCDEFGH$, trisected by two diagonals AD and EH into three quadrilaterals $ABCD$, $ADEH$ and $EFGH$ whose areas are S_1, S_2 and S_3 respectively. The area S of $ABCDEFGH$ is then

$$S = S_1 + S_2 + S_3\,. \tag{5.1}$$

For the squared areas $A_i = 16S_i^2$, $i = 1, 2, 3$ we have the formulas (4.5 ... 4.7) relying on the abbreviations (4.4).

The rationalized form of (4.1) can be written compactly as follows:

$$\Big[(A - A_1 - A_2 - A_3)^2 - 4\left(A_1A_2 + A_1A_3 + A_2A_3\right)\Big]^2 - 64AA_1A_2A_3 = 0\,. \qquad (\bigstar)$$

(This is in fact a general Brahmagupta polynomial evaluated at $a_i^2 = A_i$, $i = 1, 2, 3$, $a_4^2 = A$).
By inserting

$A_1 = 4v - u^2 + 8wy + 2uy^2 - y^4$ from (4.5),

$A_2 = 4(j + yz)^2 - (y^2 + z^2 - i)^2$ from (4.6) and

$A_3 = 4V - U^2 + 8Wz + 2Uz^2 - z^4$ from (4.7)

with

$y = |AD|, z = |EH|$

$u = a^2 + b^2 + c^2, v = a^2b^2 + a^2c^2 + b^2c^2, w = abc$

$U = e^2 + f^2 + g^2, V = e^2f^2 + e^2g^2 + f^2g^2, w = efg$

$i = d^2 + h^2, j = dh$

from (4.4) we obtain a master equation

$$\mathcal{M}(A, u, v, w, U, V, W, i, j, y, z) = 0\,.$$

Then the area equation is the discriminant of this master equation!
We conjecture similar results for arbitrary even sided cyclic polygons.

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Two reminders on Ptolemy and Ramanujan and some problems

Darko Veljan

Abstract

We present, discuss, and offer alternative proofs for a couple of beautiful results spanning almost two millennia, but unified by their connections to Indian mathematics. Several open problems are suggested for future research.

1 Introduction

The motivation for this note was the nice Croatian - Indian mathematical evening held on Dec. 20, 2024, at the Department of Mathematics, University of Zagreb, where three 20-minute lectures were given by academician Andrej Dujella, prof. dr. sc. Zvonimir Šikić and prof. dr. sc. Mirko Primc, relating their research works connected to some Indian mathematicians. The main organizer of the event was prof. dr. sc. Darko Žubrinić. After the lectures and some Indian food snacks and Croatian wines and beverages, I put on the blackboard some of Ramanujan's problems. The lecturers and other participants didn't know at the moment how to prove them (neither did I). The first topic of this note (on Ptolemy) is also deeply interconnected with Indian mathematics. So, these are the main motivations of this note. In my translation [16] of the beautiful book [11], in two topics I added some additional new stuff that does not appear in the original (as well as some others). The first is Ptolemy's formula in the topic *Ptolemy's Almagest*, year about 150 and Fuhrmann's formula, and in the year 1500 topic *The series for computing* π, I added a wonderful identity of Ramanujan which I'll explain in the sequel. This note is not only my own research, but I think it's worth reminding us of these two gems of mathematics. We end the paper with some problems (not in the original) from [16].

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2 Ptolemy's theorem

The famous mathematician and astronomer Ptolemy, or Claudius Ptolomaeus (c. 90 - c. 168) from Alexandria, published about the year 150 his comprehensive work *Almagestus*, or simply *Almagest* in 13 books, where he described almost all knowledge of astronomy and mathematics known to his time. The work is also known in Latin as *Syntaxis Mathematica*. He created a geocentric model of the Universe that was accepted as true for more than 1300 years until Copernicus' *Revolution of the Celestial Spheres* in 1543. Ptolemy had trigonometric tables of certain quantities like the function sine with measures of every 15'.
From the tables, he deduced the formula for the sine of the sum of two angles. In fact, this was the root of the theorem, many centuries later named after Ptolemy.
A (convex) quadrilateral (or any convex polygon) is called *cyclic* if it is inscribed in a circle (i.e. all of its vertices lie on a single circle). Now we can formulate the basic theorem.

Theorem 2.1 (Ptolemy's theorem (about AD 150))**.** *A quadrilateral $ABCD$ (vertices in this order) is cyclic if and only if the product of the lengths of its diagonals is equal to the sum of the products of the lengths of the pairs of opposite sides.*

In symbols, if $|XY|$ is the length of the segment between points X and Y, and if we denote $|AB| = a$, $|BC| = b$, $|CD| = c$, $|DA| = d$, $|AC| = e$, $|BD| = f$, then we have:

$$ef = ac + bd. \tag{2.1}$$

It seems that the first rigorous proof of this theorem was given by the Arab mathematician (and translator) Abul Wafa (or Wefa) about AD 980. However, many used the Ptolemy formula much earlier. For example, the Indian mathematician Brahmagupta (598-660) used Ptolemy's theorem to compute the area and the radius of the cyclic quadrilateral in terms of side lengths around year 650. In fact, Brahmagupta first proved (with the same notation as above) that

$$\frac{f}{e} = \frac{ad + bc}{ab + cd}. \tag{2.2}$$

Equations (2.1) and (2.2) enable us to express the lengths of diagonals in terms of side lengths of cyclic quadrilaterals. Then, using the well-known Heron's formula from about AD 60, which gives the triangle area in terms of its side lengths, Brahmagupta computed the area S and the radius R of the cyclic quadrilateral in terms of its side lengths a, b, c, d as

$$16S^2 = (a + b + c - d)(a + b - c + d)(a - b + c + d)(-a + b + c + d), \tag{2.3}$$

and

$$\begin{aligned} R^2(a+b+c-d)(a+b-c+d)(a-b+c+d)(-a+b+c+d) = \\ = (ab+cd)(ac+bd)(ad+bc). \end{aligned} \tag{2.4}$$

Note that Brahmaguptas's formulae reduce to triangle formulae for say, $d = 0$. More precisely, formula (2.3) reduces to Heron's formula and formula (2.4) to the triangle's circumradius. Apparently, some of these formulae for triangles were known even to Archimedes about 220 B.C.

Proof of Ptolemy's theorem. There are many known proofs of Theorem 2.1, as presented e.g. in [1]. Perhaps the shortest and most elegant proof is by inversion. Choose a big circle K with the center, say, D and radius r, so that the circumcircle $k = ABCD$ is inside K. Consider the inversion $I = I(D, r)$. Then k is mapped into a line k'. Let $A' = I(A)$, $B' = I(B)$, $C' = I(C)$. Then $|A'B'| + |B'C'| = |A'C'|$ on the line k'. But

$$|A'B'| = \frac{|AB|r^2}{|DA||DB|},$$

and similarly for $|B'C'|$ and $|A'C'|$. So, equality (2.1) follows.
Conversely, if one of the vertices does not lie on the circle k, say, B, then $|A'B'| + |B'C'| > |A'C'|$, by triangle inequality, hence $ac + bd > ef$.
The formula (2.2) can also be easily proved via inversion. For a more general fact see *Mathologer, Ptolemy's theorem.* □

Ptolemy's theorem is equivalent to the following facts: the addition formulas for sine and cosine, Pythagoras' theorem, the sine law for triangles, the cosine law for triangles, and many more. Since Pythagoras' theorem is equivalent to Euclid's fifth postulate, we may say that Ptolemy's theorem is in the essence of Euclidean geometry.
Today there are many generalizations, extensions, corollaries, and equivalent statements of Ptolemy's theorem beyond those already mentioned. Even some Croatian mathematicians contributed to the topic, e.g. [4, 8, 9]. One of the best-known generalizations and most quoted extensions of Ptolemy's is Fuhrmann's hexagon theorem which I also quoted in [16] in the topic *Ptolemy's Almagest* (150). This theorem (see [2]) is named after the German mathematician Wilhelm Fuhrmann (1833-1904).

Theorem 2.2 (Fuhrmann's theorem (1890))**.** *Let the opposite side lengths of a convex cyclic hexagon be a, a', b, b' and c, c', and let e, f, g be the polygon (big) diagonals, such that a, a' and e have no common polygon vertex, and likewise for b, b' and f and c, c' and g. Then*

$$efg = aa'e + bb'f + cc'g + abc + a'b'c'. \tag{2.5}$$

Idea of the proof. Let $ABCDEF$ be the cyclic hexagon with side lengths $AB = a$, $BC = b$, $CD = c$, $DE = a'$, $EF = b'$, $FA = c'$ and $CF = e$, $DA = f$ and $BE = g$. Here $AB = |AB|$, etc. We apply Ptolemy's theorem to each of the four convex cyclic quadrilaterals $ABDE$, $BCDF$, $ADEF$, and $ABEF$. After some simple algebraic manipulations with Ptolemy's relations, we can obtain the formula (2.5). We omit some tedious computational details. See also some Internet sites such as [2]. □

Ptolemy's theorem in the hyperbolic plane, say with curvature -1, is given by the following. The formula is the same as (2.1), but instead of x now we have $s(x) = \sinh\left(\frac{x}{2}\right)$.

Theorem 2.3 (Ptolemy's theorem in hyperbolic geometry)**.** *Let $ABCD$ be a convex hyperbolic quadrilateral inscribed in a hyperbolic circle. Then*

$$s(|AC|)\, s(|BD|) = s(|AB|)\, s(|CD|) + s(|AD|)\, s(|BC|)\,. \tag{2.6}$$

The converse is also true. A convex hyperbolic quadrilateral $ABCD$ has a hyperbolic circumcircle if three of the points lie on a hyperbolic circle and satisfy equation (2.6).

Proofs are similar to the original proof of Ptolemy's theorem and can be found in the literature cited before. Of course, the spherical version of Ptolemy holds as well with the same formula (on the unit sphere) with $s(x) = \sin\left(\frac{x}{2}\right)$.
In [14], we managed to prove some interesting geometric facts on cyclic pentagons and, among other things, we proved the Robbins formulae which gives a polynomial equation for the area and radius of a cyclic pentagon in terms of its side lengths, something like Brahmagupta's formulas (2.3) for cyclic quadrilateral, but much more involved. A nice survey on the topic of Robbin's conjectures is given in [10]. Recently, D. Svrtan in [13] used Hopf-Wiener factorization of certain Laurent polynomial invariant of cyclic polygons and by tricky computer search obtained huge polynomials for cyclic n-gons areas and circumradius for $n = 4, 5, 6, 7$ and 8.
As said earlier, there are many generalizations of Ptolemy's theorem. The best seems to me is the following from [3].

Theorem 2.4 (M. Bencze, 2011)**.** *Let $A_1A_2,\dots A_n$ be a convex cyclic Euclidean polygon with vertices in given order. Then the following holds*

$$\frac{|A_2A_n|}{|A_1A_2||A_1A_n|} = \frac{|A_2A_3|}{|A_1A_2||A_1A_3|} + \frac{|A_3A_4|}{|A_1A_3||A_1A_4|} + \cdots + \frac{|A_{n-1}A_n|}{|A_1A_{n-1}||A_1A_n|}. \tag{2.7}$$

Proof is again by inversion. Note that for $n = 4$ we get Ptolemy's relation (2.1), for $n = 5$ we have this in [14] in some form, and for $n = 6$ we get Fuhrmann's formula (2.5).
A generalization of Ptolemy's theorem in n-dimensional Euclidean space was given in [5]. Furthermore, a very recent analog of Fuhrmann's theorem in the Lobachevsky plane was given in [7]. Here we stop on Ptolemy.

3 Some Ramanujan identities and conjectures

Now we shall consider a completely different topic, but also deeply connected to Indian mathematics. It is about the brilliant Indian mathematician Srinivasa Ramanujan (1887-1920), see his Collected papers and problems with some solutions [6] having 355 pages (which my colleague M. Primc kindly lent me after our Croatian-Indian math evening).
In [16], I put the following Ramanujan's identity in the article *Series for Computing* π, the year 1500:

$$A + B = \sqrt{\frac{\pi e}{2}}. \tag{3.1}$$

Here A is the infinite series

$$A = 1 + \frac{1}{1 \cdot 3} + \frac{1}{1 \cdot 3 \cdot 5} + \cdots = \sum_{n \geq 1} \frac{1}{(2n-1)!!},$$

and B is the infinite continuous fraction

$$B = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{1 + \cfrac{3}{1 + \cdots}}}}.$$

The exact values of both A and B are not known, but still, their sum is the square root of $\frac{\pi e}{2}$. I have seen it somewhere and couldn't resist but put that gem in my translation [16]. Now I found it in [6], p. 341, as Ramanujan's question 541 in the Indian Journal of Mathematics from 1914. I tried to prove it but with no success. There is no solution in [6]. Then the organizer of that event, my colleague D. Žubrinić, sent me the link of Mathologer https://www.youtube.com/watch?v=6iTdNmDHfV0 with a very nice explanation and proof of formula (3.1).
So, following this link, I'll try to present proof of 3.1. Once more this proof was shown on *Mathologer Masterclass* on the above link under the title *Ramanujan's easiest hard infinite monster* on June 24, 2023.

Proof of Ramanujan identity (3.1). First, recall the *Gauss normal distribution integral formula.* The area under the Gauss bell is

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-t^2} dt \tag{3.2}$$

See e.g. Wikipedia on *Gaussian integral.* On the other hand, recall *Wallis' formula* from 1665:

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdots}. \tag{3.3}$$

A short proof of Wallis' formula is as follows. It is well known that

$$\sin x = x \prod_{k \geq 1} \left(1 - \left(\frac{x}{k\pi}\right)^2\right).$$

Substituting x with $\frac{\pi}{2}$ yields formula (3.3). This is a special case of *Euler's product formula*

$$\frac{\sin(\pi z)}{\pi z} = \prod_{n \geq 1} \left(1 - \frac{z^2}{n^2}\right),$$

valid for any complex number z. It is also known as *Euler's sinc function formula* (see e.g., https://proofwiki.org/wiki/Euler_Formula_for_Sine_Function/Complex_Numbers).

Now consider the following series to get A in (3.1). Let

$$y(x) = \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n \geq 1} \frac{x^n}{n!}. \tag{3.4}$$

By taking derivative of (3.4), we have

$$y'(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = 1 + \sum_{n \geq 1} \frac{x^n}{n!}. \tag{3.5}$$

Hence, $y(x) = Ce^x - 1$, but from $y(0) = 0$, we get $C = 1$, so $y(x) = e^x - 1$. So, for $x = 1$, we have

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots,$$

i.e. the well-known Euler's number e, and the well-known series expansion of the exponential function

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

Now consider the function

$$y(x) = \frac{x}{1} + \frac{x^3}{1 \cdot 3} + \frac{x^5}{1 \cdot 3 \cdot 5} + \frac{x^7}{1 \cdot 3 \cdot 5 \cdot 7} + \cdots = \sum_{n \geq 1} \frac{x^{2n-1}}{(2n-1)!!}. \tag{3.6}$$

By taking the derivative of (3.6), we get the differential equation

$$y'(x) = 1 + xy(x), \tag{3.7}$$

with $y(0) = 0$. The solution of this linear ordinary differential equation of the first order (3.7), as known from the theory of ODE, is given by

$$y(x) = e^{\frac{x^2}{2}} \int_0^x e^{-\frac{t^2}{2}} dt. \tag{3.8}$$

So, the right-hand side of (3.8) is the right-hand side of (3.6). So far we know that

$$\sqrt{\frac{\pi e}{2}} = A + \sqrt{e} \int_1^\infty e^{-\frac{t^2}{2}} dt. \tag{3.9}$$

What is left to prove is that the second summand on the right-hand side of (3.9) is equal to the continuous fraction B from (3.1). Now consider the function

$$y(x) = e^{\frac{x^2}{2}} \sqrt{\frac{\pi}{2}} - e^{\frac{x^2}{2}} \int_0^x e^{-\frac{t^2}{2}} dt. \tag{3.10}$$

By taking the derivative of (3.10), it is easy to check that we get the differential equation (a bit different from (3.7)):

$$y'(x) = xy(x) - 1, \tag{3.11}$$

with $y(0) = \sqrt{\frac{\pi}{2}}$. Keep taking the derivatives of (3.11) repeatedly, we obtain

$$\begin{aligned} y' &= xy - 1 \\ y'' &= xy' + y \\ y''' &= xy'' + 2y \\ y'''' &= xy''' + 3y'' \\ &\vdots \end{aligned}$$

Hence $\frac{y'}{y} = x - \frac{1}{y}$, $\frac{y''}{y'} = x + \frac{y}{y'}$, $\frac{y'''}{y''} = x + 2\frac{y'}{y''}$, $\frac{y''''}{y'''} = x + \frac{3y''}{y'''}$, and so on. Then by substituting and by little calculation we finally get

$$y(x) = \cfrac{1}{x + \cfrac{1}{x + \cfrac{2}{x + \cfrac{3}{x + \cfrac{4}{\ddots}}}}}. \tag{3.12}$$

Setting $x = 0$ in (3.12) we get

$$y(x) = \cfrac{1}{\cfrac{1}{\cfrac{2}{\cfrac{3}{\cfrac{4}{\ddots}}}}},$$

and by taking little care of this infinite fraction we obtain that it is equal to

$$\frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdots}.$$

By Wallis' formula, this is equal to $\frac{\pi}{2}$. Finally, we therefore proved (with some care on convergence) that for all $x > 0$, we have

$$e^{\frac{x^2}{2}}\sqrt{\frac{\pi}{2}} = \left(\frac{x^1}{1} + \frac{x^3}{1 \cdot 3} + \frac{x^5}{1 \cdot 3 \cdot 5} + \cdots\right) + \cfrac{1}{x + \cfrac{1}{x + \cfrac{2}{x + \cfrac{3}{x + \cfrac{4}{\ddots}}}}}. \tag{3.13}$$

Thus, (3.13) holds also for $x = 1$. Thus, the identity (3.1) is proved. □

In the Collected Papers [6] of S. Ramanujan there are many interesting theorems, identities, approximations, formulas, and conjectures. For instance, Ramanujan proved that for a sufficiently big natural number n, there are, as a rule, $\log\log n$ prime divisors of n. The next example is his approximation $\pi \approx \frac{63(17+15\sqrt{5})}{25(7+15\sqrt{5})}$ which is exactly up to 9 decimals (of course, without computers!). Even today, certain aspects of the so-called „combinatorial Rogers-Ramanujan identities "are the topic of current research, e.g. see [12] by Croatian mathematician Mirko Primc, an expert in applications of representation theory and Lie algebra theory in combinatorics.
Here is one of Ramanujan's problems posed in 1913 (from [6]):
Compute

a)

$$\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \cdots}}}, \tag{3.14}$$

b)

$$\sqrt{6 + 2\sqrt{7 + 3\sqrt{8 + \cdots}}}. \tag{3.15}$$

Solution by S. Ramanujan.
a) It holds $n(n+2) = n\sqrt{1+(n+1)(n+3)}$. Let $f(n) = n(n+2)$. Then

$$f(n) = n\sqrt{1+f(n+1)} = n\sqrt{1+(n+1)\sqrt{1+f(n+2)}} = \cdots ,$$

hence

$$n(n+2) = n\sqrt{1+(n+1)\sqrt{1+(n+2)\sqrt{1+\cdots}}}.$$

For $n = 1$, the result of a) is equal to 3.
b) Similarly, let $f(n) = n(n+3)$. Since $n(n+3) = n\sqrt{n+5+(n+1)(n+4)}$, we have

$$\begin{aligned} f(n) &= n\sqrt{n+5+f(n+1)} \\ &= n\sqrt{n+5+(n+1)\sqrt{n+6+f(n+2)}} = \cdots , \end{aligned}$$

and for $n = 1$, we get that the result of b) is equal to 4.
Ramanujan also conjectured many identities and a lot of claims which he or other people resolved later. One easy is that the number $0.2357111317192329\ldots$ (concatenation of all primes after the decimal comma) is not a rational number. But his conjecture that the number $\pi + e$ is not rational is still not resolved. He knew Leibniz's formula $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$, which follows from the series expansion $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ for $x = 1$, and Euler's number $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$ and was not sure about their sum, although both π and e are transcendental. Also, an open problem is that 2^e is not rational. It was only in 1934 that was proved that 2 to the power of $\sqrt{2}$ and $\frac{\log 2}{\log 3}$ are transcendental. It is quite possible that Ramanujan considered such problems much earlier.
Apparently, Leibniz's formula for $\frac{\pi}{4}$ was known to the Indian mathematician Nilakanthi Somayai (1444-1530) and also to Scott James Gregory (1638-1675). Leibnitz proved it in 1673 geometrically. Even Newton praised that formula and said that it showed that Leibniz was a genius, although they had a long dispute about whose contribution to calculus was most influential. In any case, Srinivasa Ramanujan definitely was a genius.
In the end, we provide some questions, sayings, and open problems I put in [16].
Besides Millenium problems which I included in my translation (not in the original), I also added some solved and unsolved problems in the translation [16] of [11].
Here are some.

- Four girls are bathing in the (say) Adriatic sea. Each two are at a distance from each other at about 25 meters. Three of the girls have red bikinis, what has the fourth girl on herself?

- (Paul Erdös,1936) If a set A of natural numbers has the property that the sum of the reciprocals diverges, then A has an arithmetical sequence of arbitrary length (true for A primes, by T. Tao and B. Green, 2012)

- David Hilbert (1862 -1943) once said that if he awakens (in some sense) 1000 years from now, his first question would be: Is the Riemann hypothesis solved?

- Euler's perfect brick (or box) problem (about 1772): Is there a perfect brick? A perfect brick is a quadrum (brick) with all lengths of edges, diagonals (plane and space) are whole numbers.

- Graham's problem (1996): Is the sequence (a_n), unbounded if $a_0 = 2$ and $a_{n+1} = a_n - \frac{1}{a_n}$?

- Geometry problem (from 1930th): Which polyhedron on n vertices on the unit sphere has the maximal volume? (The five Plato's bodies are solutions, but in general?)

- Atiyah's conjecture (1998) on star configurations: Consider $n > 2$ points („stars “) in space, not all on a line. From any point („star “) consider $n-1$ directions to other „stars “(considered as complex numbers on the unit sphere). Attach to any point („star “) the polynomial whose directions are the roots. Then is the set of these n polynomials linearly independent (over the complex numbers)?

- (D. Veljan, 2023): The probability that a randomly and uniformly chosen point from the circumball of a tetrahedron is out of the inscribed ball is greater than or equal to $1 - \sqrt{\frac{d_3}{(3e_1)^3}}$, (see [15]) where $e_1 = aa' + bb' + cc'$, $d_3 = (aa' + bb' - cc')(aa' - bb' + cc')(-aa' + bb' + cc')$, and a and a' are the opposite side lengths of a tetrahedron and similarly for b, b' and c, c'. (We can think of vertices of the tetrahedron as stars and the chosen point as an exoplanet.) What are the hyperbolic 3D and 4D versions of this fact with respect of the complexity of our Universe?

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Beauty of the Canvas Aspect Ratios 1.357 and 1.441

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Abstract

Recently, a collection of more than 223 thousand paintings have been analyzed and it was established that the average aspect ratio for portraits is 1.357:1, and for landscape-oriented paintings is (close to) 1.441:1. Using the wisdom of the crowd theory, these two numbers should be related to some universal beauty that surpasses individual personal preferences. We show that indeed these values are related to important mathematical proportions (arithmetical mean, Kepler triangle, golden section) and that the difference between aspect ratios of vertically and horizontally oriented paintings is related to the peripheral vision field. These aspect ratios can be used by painters and frame manufacturers to amplify the beauty of artistic compositions taking into consideration the psychology of perception –our ability to innocuously register proportion as beauty. Very few real numbers are so special, that they should be widely known in the artistic world (e.g. golden ratio). It might be that these two numbers could deserve such status.

1 Introduction

The goal of this paper is to try to determine if there is an optimal canvas aspect ratio. This problem is closely related to the long-standing problem of determining if there is aesthetically the most pleasing aspect ratio of the rectangle sides. Fechner [9] introduced three ways to approach this problem almost 150 years ago [13]:

1) “the method of choice (Wahl), in which subjects choose, from among a number of alternatives, the item that they like (or dislike) the most;

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2) the method of production (Herstellung), in which subjects are asked to draw, or otherwise create, an object of a certain kind that has features or proportions they find most agreeable (or disagreeable);

3) the method of use (Verwendung), in which the experimenter examines preexisting objects of the kind being studied, and determines whether they conform to certain hypotheses about the determination of aesthetic pleasure.

In the same paper, Fechner concluded that the most beautiful rectangle is a rectangle with the aspect ratio of its sides equal to the golden ratio - two quantities are in the golden ratio if their ratio is the same as the ratio of their sum to the larger of the two quantities. It is denoted by φ in honor of Greek sculptor Phidias (480-430 BC), painter and architect in whose artworks lots of instances of the golden section have been detected and it is equal to $\varphi = \frac{\sqrt{5}+1}{2} \approx 1.618$. Euclid (300BC) started to study its mathematical properties and since antiquity, this number has attracted scientists and artists –it appears in the abundance of natural phenomena and many artworks incorporate it [5, 16, 17].
Fechner' s observation will steer up quite a controversy. Very early, Scripture [22] and Woodworth' s [28] interpretation of the results of Thorndike [26] strongly supported Fechner' s findings (nice illustrative graphs of both findings can be seen as Figure 2 and Figure 3 in [3]. Throughout the years many more scientists also supported this result [4, 15, 18, 21]. Partial support and partial opposition to this finding can be found in the paper [7] where Fechner' s observation was concurred only for introverts, but disputed for extroverts.
On the other hand, strong opposition to these results can be found in the papers of Godkewitsch [12] and Green [14] which provided strong arguments that the methodology of the previous research had some flaws. Russel [20] finds the average, median, and mode of preferred aspect ratios of experimental subjects all different from the golden section.
As a summarized conclusion of previous research, one may cite Green [13]: "I am led to the judgment that the traditional aesthetic effects of the golden section may well be real, but that if they are, they are fragile as well. Repeated efforts to show them to be illusory have, in many instances, been followed up by efforts that have restored them, even when taking the latest round of criticism into account."
Researchers of the most beautiful aspect ratio for the rectangle almost exclusively used the first two methods that Fechner proposed (Wahl and Herstellung). In this paper, we will analyze what can be learned from the third method (Verwendung). We will compare the results of these findings with

the mathematically indicated ideal aspect ratio of canvas and obtain an almost perfect match.

2 Methods

2.1 Wisdom of the crowd and beyond

Jeroen Visser analyzed a collection of more than 223 thousand paintings: 112476 portraits and 110611 landscape-oriented paintings in his master thesis [27]. He obtained that the average aspect ratio of a portrait is (height to width) 1.357:1 and that the average ratio of a landscape-oriented painting is (width to height) 1.45:1.

At first sight, these two numbers do not have any obvious significance. E.g. the only ratio offered for canvas prints by Saatchi art are $\frac{1}{1} = 1$, $\frac{5}{4} = 1.25$, $\frac{4}{3} \approx 1.333$ and $\frac{3}{2} = 1.5$, none of which is too close to these two numbers.

However, the Wisdom of the crowds theory suggests differently. This theory starts with the famous Francis Galton observation of a cow-weight guessing contest [11] where the average guess of cow's weight was within 0.8% of cow's weight although individual guesses were mostly quite different from correct weight. The basis of this theory is the law of large numbers first discovered by Cardano in the 16th century which implies that if errors of individual guesses are bounded and independent, then the error of the average will be extremely small (for simple proof see [6] and for more details about this theory see [25] and references within).

Suppose that a human's sense of beauty comprises individual preference and the objective concept of beauty. Applying the same methodology as Galton did –i.e. averaging the senses of beauty of multiple individuals, one might be able to distill an objective concept of beauty. If so, then ratios 1.357 : 1 and 1.45 : 1 (or ratios very close to these numbers) might have some special status.

Let us note that in fact, the analysis of the average aspect ratio of paintings goes beyond the wisdom of crowds. Namely, the reasonably small price of the ticket in Galton's experiment is quite different from producing a painting in which the artist invests considerable time and effort. Hence, instead of guessing, we could say that painters bet big time on the aspect ratios (among many different painting elements) and experiments where significant betting is included give even better results than simple averaging. There is an old saying "Put your money where your mouth is" (and it is used in the title of the paper Fang, Stinchcombe, and Whinston [8] that analyzes such phenomena).

2.2 Mathematical analysis

When observing a rectangle, there are three lengths that one may observe: each of two sides and a diagonal. Let us denote the smaller side by s, the larger by l, and the diagonal by d. Two most simple regularities that three numbers can show is that the middle one is either the arithmetic mean (average) or the geometric mean of the other two, i.e. that

$$l = \frac{s+d}{2} \text{ or } l = \sqrt{sd}.$$

Considering that the Pythagorean theorem implies that $d^2 = l^2 + s^2$, simple calculation shows that arithmetic mean implies that

$$s : l : d = 3 : 4 : 5.$$

This is the smallest Pythagorean triple (a reader interested in Pythagorean triples is referred to [23]. Calculation using geometric mean implies that

$$s : l : d = 1 : \sqrt{\varphi} : \varphi,$$

where $\varphi = \frac{\sqrt{5}+1}{2} \approx 1.618$ is the golden ratio. The triangle with side ratios $1 : \sqrt{\varphi} : \varphi$ is called the Kepler triangle. Kepler was fascinated by this peculiar connection of golden ratio and Pythagorean theorem stating that: "Geometry has two great treasures: one is the theorem of Pythagoras, the other the division of a line into extreme and mean ratio. The first we may compare to a mass of gold, the second we may call a precious jewel" [10].
Hence, we have two important rectangles –one with the side ratio $\sqrt{\varphi} \approx 1.272$ and the other with the side ratio $\frac{4}{3} \approx 1.333$. Note that when an observer faces a piece of art, he does not face unframed canvas, but framed canvas. Hence, one might wonder what should be the ratio of unframed canvas that would produce a ratio of framed canvas 1.272 and 1.333. Obviously, this depends on the width of the frame. Hence, one might ask if the canvas is given is there some method of calculating the optimal width of the frame?
One of the possible ways is to put such a frame that incorporates nice proportions. We have three areas: unframed canvas area (let us denote it by u), frame area (let us denote it by f), and total framed painting area (which is equal to $u + f$). Hence, it can be required that

$$u : f = (u + f) : u.$$

Then, $(u + f) : u$ is the golden ratio. Such choice of frame width is already advised by many makers of custom-made frames (e.g. see [1, 2]). Let us call such framing golden framing. Let us define the function $g : \langle 1, +\infty\rangle \to$

$\langle 1, +\infty\rangle$ that we call the golden lift. Its input argument is ratio x of the longer side of a framed picture to its shorter side and its result is the ratio $g(x)$ of the longer side of the canvas to the shorter when golden framing is applied. Let us denote the width of the frame by $w(x)$. It holds that

$$[g(x) + 2w(x)] \cdot [1 + 2w(x)] = \varphi\,[g(x) \cdot 1]\,; \tag{2.1}$$

$$[g(x) + 2w(x)] : [1 + 2w(x)] = x. \tag{2.2}$$

From (2.2), we get

$$w(x) = \frac{g(x) - w}{2x - 2}.$$

Inserting in (2.1), we get

$$\left(g(x) + 2\frac{g(x) - x}{2x - 2}\right)\left(1 + 2\frac{g(x) - x}{2x - 2}\right) = \varphi \cdot g(x)$$

which is equivalent to:

$$(g(x))^2 - \left(2 + \varphi x + \frac{\varphi}{x} - 2\varphi\right) g(x) + 1 = 0.$$

Solving it for $g(x)$, we get:

$$g(x) = \frac{2 + \varphi x + \frac{\varphi}{x} - 2\varphi \pm \sqrt{\left(2 + \varphi x + \frac{\varphi}{x} - 2\varphi\right)^2 - 4}}{2}.$$

The solution with the minus sign gives $g(x) < 1$ which is incorrect. Hence,

$$g(x) = \frac{2 + \varphi x + \frac{\varphi}{x} - 2\varphi + \sqrt{\left(2 + \varphi x + \frac{\varphi}{x} - 2\varphi\right)^2 - 4}}{2}.$$

Now we have $g(\sqrt{\varphi}) \approx 1.357$ and $g(\frac{4}{3}) \approx 1.441$.

3 Results and discussion

Note that $g(\sqrt{\varphi})$ coincides in all four digits with the average ratio of the longer to shorter side of the canvas for portraits calculated in [27]. This kind of agreement can hardly be accidental. Hence, two completely different approaches provide the same result. This number $g(\frac{4}{3})$ is 0.6% less than the average of the ratio of longer to shorter side for all landscape-oriented paintings. Hence, there is almost a perfect match. One reason for the small discrepancy is that the aspect ratios of the painting are sometimes divided into three groups (portrait, square, landscape), and sometimes in four groups (portrait, square, landscape, and panoramic). Hence, the wisdom of

the crowd might not imply the result of 1.45, but slightly less than 1.45. This would be in accordance with the calculated value of $g(\frac{4}{3}) \approx 1.441$.
The remaining interesting question is why there is a difference between the average ratio of portraits and landscape-oriented paintings and the answer might be rooted in the shape of the peripheral visual field. Namely, the peripheral visual field is horizontally elongated (see Figure 6 in [19] or detailed review [24] and references within). Hence, for landscape orientation, an obvious choice is the larger of these two possibilities (hence indeed: 1.441), and for portrait smaller of these values (hence indeed: 1.357).

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Maximal matchings in multiple ring networks with shared link

Ivana Zubac

Abstract

Graph theory is an extremely diverse field with wide applications today. Graphs have proven to be an excellent tool for modeling systems, emphasizing connections and relationships between objects. In graph theory, matching is a fundamental concept used to describe a set of edges without common vertices. Understanding them is essential for solving problems involving efficient routing and resource allocation. In this work, we enumerate maximal matchings and determine the saturation and matching number in book graphs, which are suitable for representing certain configurations of computer networks.

Keywords: cycle related graphs, book graph, maximal matching, saturation number, matching number.
2020 *Mathematics Subject Classification*: 05C70, 05C38, 68R10.

1 Introduction

Graph theory is an extremely diverse field with wide applications today. Graphs have proven to be an excellent tool for modeling systems emphasizing connections and relationships between objects. If we pay attention, we will notice that the problems studied by graph theory are everywhere around us. In this paper, we will show the properties of book graphs that are inspired by a type of network topology.

A graph $G(V, E)$ is a pair of two sets, V and E, $V = V(G)$ being a finite nonempty set and $E = E(G)$ is a binary relation defined on V. A graph can be visualized by representing the elements of V by points (vertices) and

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joining pairs of vertices (i, j) by an edge (bond) if and only if $(i, j) \in E(G)$. The number of vertices in G equals the cardinality $n = |V(G)|$ of this set. The degree of a node in a non-directed graph is defined as the number of links a node has with other nodes.
We assume that the reader is familiar with basic graph-theoretic concepts such as degree, neighborhood, etc., and with basic classes of graphs such as paths, cycles, and complete graphs. Here we denote by P_n a path on n edges ($n + 1$ vertices) and by C_n a cycle on n vertices. All graphs in this paper are finite, simple, and undirected. Terms not defined here are used in the sense of Harary [4].

In graph theory, *matching* is a fundamental concept used to describe a set of edges without common vertices. Matchings are used in various applications such as network design (efficient routing and resource allocation), job assignments (assigning jobs to machines or workers), scheduling (optimal scheduling of tasks), chemistry, graph coloring, neural networks in artificial intelligence, and more. The cardinality of M is called the size of the matching. As the matchings of small sizes are not interesting, we will be mostly interested in matchings that are as large as possible. A matching M is *maximum* if there is no matching in G with more edges than M. The cardinality of any maximum matching in G is called the *matching number* of G and denoted by $\nu(G)$. Since each vertex can be incident to at most one edge of a matching, it follows that the matching number of a graph on n vertices cannot exceed $\left\lfloor \dfrac{n}{2} \right\rfloor$.
The matching M is *perfect* if each vertex of G is incident with an edge of M. Perfect matchings (also known in chemistry as Kekulé structures) are also maximum matchings [5].
There is another way to quantify the idea of "large" matching. A matching M in G is *maximal* if no other matching in G contains it as a proper subset. Obviously, every maximum matching is also maximal, but the opposite is generally not true. Maximal matchings are much less researched with respect to both their structural and enumerative properties. Maximal matchings can serve as models of several technical problems such as the block-allocation of a sequential resource. The cardinality of any smallest maximal matching in G is the *saturation number* of G. The saturation number of a graph G we denote by $s(G)$. It is easy to see that the saturation number of a graph G is at least one-half of the matching number of G, i.e., $s(G) \geq \nu(G)/2$. Hence, the saturation number provides a piece of information on the worst possible case.

Network topology refers to the arrangement and interconnection of various components within a (computer) network, including nodes (computers,

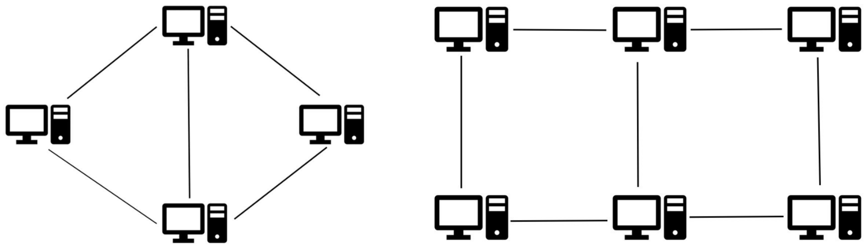

Figure 1: Multiple Ring Networks with Shared Link

switches, routers) and links (wired or wireless connections). It defines how these components are connected and interact with each other. Physical topology refers to the placement of the network's various components, including the device locations and cable installation, while logical topology shows how data flows within the network, regardless of its physical design [3].
The structure of a network topology determines how data is transmitted, affecting the network's performance, reliability, and scalability. An efficiently designed topology can reduce cable costs, enhance data transfer speeds, and improve network reliability. On the other hand, a poorly thought-out topology can lead to congested data paths and increased risk of network failures. For organizations, choosing the right topology is a key part of network planning, as it affects both the operational efficiency and the ease of future expansion. The landscape of network topology is diverse, offering various configurations, each with its unique characteristics and suitability for different network scenarios.

The primary types of network topology include: Point-to-Point (represented by path graph), Bus Topology (caterpillar), Star (star graph), Ring (cycle graph), Tree (tree graph), Mesh (with each node having a connection to several other nodes), Hybrid (combines two or more different types of topologies).
In this paper, we are concerned with graphs representing one possible network topology, the Multiple Ring Network with Shared Link. The shape of that topology can be represented by a graph we call a book graph. We represent the components, nodes (computers, switches, routers) by vertices and links (wired or wireless connections) by edges of certain graphs. Book graphs consist of a certain number of cycles, not necessarily of the same length, which all share one common edge. The cycle lengths are at least three. See examples in Fig.2.

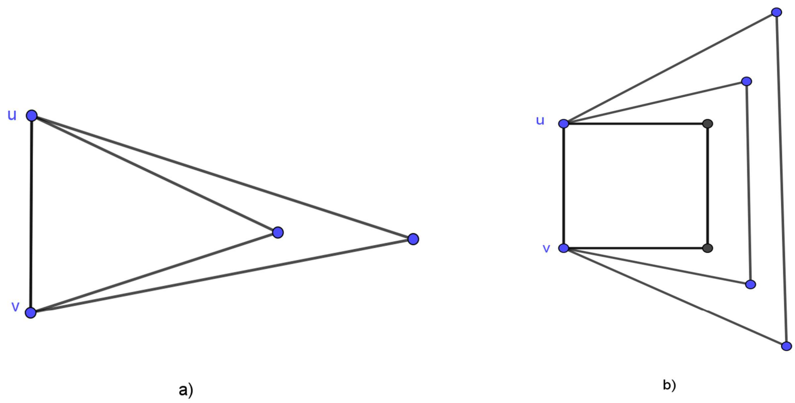

Figure 2: A book graph a) $B(2,1)$ with 2 sheets b) $B(3,2)$ with 3 sheets

2 Maximal matchings in book graphs

In this section, we state and prove our main results about the number of maximal matchings in book graphs. We refer the reader to [4] for all graph-theoretical terms not defined here.

A *book graph* $B = B(n,k)$ is a graph with $nk+2$ vertices, consisting of n cycles C_{k+2}, that all share exactly one common edge. Let us denote the vertices of the common edge with u and w. The other vertices we denote with $v_{11}, ..., v_{1k}, v_{21}, ..., v_{2k}, ...v_{n1}, ..., v_{nk}$, where the first label, say m, indicates the cycle, and the second label indicates the position of a vertex in the m-th cycle. In all cycles, the second vertex labels are increasing when proceeding along the cycle from u to w. See Figure 3. In order to avoid problems with too few, or with too short cycles, we restrict our attention to $n \geq 2$ and $k \geq 3$. We denote the number of maximal matchings in $B(n,k)$ by $\Psi(n,k)$.
First, we settle the two shortest cases, $k = 1$ and $k = 2$.

Lemma 2.1. *Let $n \geq 2$. Then the number of maximal matchings in $B(n,1)$ is given by*

$$\Psi\,(n,1) = n(n-1) + 1, \tag{2.1}$$

and the number of maximal matchings in $B(n,2)$ is given as

$$\Psi\,(n,2) = n^2 + 1. \tag{2.2}$$

Proof. We start with an observation, valid also for $k \geq 3$, that any maximal matching must cover at least one of the vertices u and w. The number of

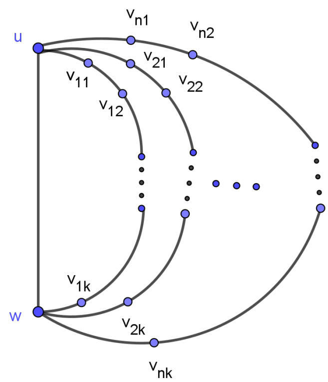

Figure 3: A book graph $B(n,k)$

maximal matchings in $B(n,1)$ in which both u and w are covered by the same edge, hence uw, is exactly one. If u and w are covered by different edges, the edge covering u can be chosen in n ways, leaving $n-1$ possibilities to choose the edge covering w, since those edges cannot belong to the same cycle. Since it is not possible to have either one of u and w uncovered by a maximal matching, we have exhausted all possibilities and the total number of maximal matchings in $B(n,1)$ is equal to $n(n-1)+1$, as claimed.
The number of maximal matchings in $B(n,2)$ covering both u and w by the same edge is again 1. If those vertices are covered by different edges, there are $n \cdot n = n^2$ such possibilities. Again, it is not possible to have just one of them covered by a maximal matching, a consequence of random matchability of cycles C_4 making the sheets of the considered books. Hence, $\Psi(n,2) = n^2 + 1$, as claimed in the statement. □

Both sequences $\Psi(n,1)$ and $\Psi(n,2)$ appear in the *On-Line Encyclopedia of Integer Sequences* [1], $\Psi(n,1)$ as A002522, and $\Psi(n,2)$ as A002061. Both have a number of other combinatorial interpretations, but maximal matchings are not among them. It would be an interesting exercise to construct explicit bijections between some of those interpretations and our maximal matchings.

Before we consider the general case, we notice for that any maximal matching in $B(n,k)$ covering both u and w, the remaining graph decomposes into a disjoint union of paths of the same, or almost same, length. Hence, we quote a result on the number of maximal matchings in paths [2].

Proposition 2.2 ([2]). *Let ψ_k denote the number of maximal matchings in P_k. The sequence ψ_k satisfies the linear recurrence*

$$\psi_k = \psi_{k-2} + \psi_{k-3},$$

with the initial conditions $\psi_0 = \psi_1 = \psi_2 = 1$.

The enumerating sequence of the number of maximal matchings in P_k is the (shifted) Padovan sequence, sequence A000931 from [1]. We invite the reader to refer to OEIS for other combinatorial representation of this sequence.
The above lemma will be useful also for the case when only one of u and w is covered by a maximal matching. In that case, all neighbors of the other one must be covered, and the graph again decomposes into several disjoint paths, their length again quite similar.

Proposition 2.3. *The sequence $\Psi(n,k)$ is given by*

$$\Psi(n,k) = \psi_k^n + n\,\psi_{k-2}\,\psi_k^{n-1} + n(n-1)\,\psi_{k-1}^2\,\psi_k^{n-2} + 2\,n(n-1)\,\psi_{k-3}\,\psi_{k-2}^{n-1},$$

for $n \geq 2, k \geq 3$, where ψ_k denotes the number of maximal matchings in P_k.

Proof. As mentioned before, any maximal matching in $B(n,k)$ must cover at least one of the vertices u and w. We first look at the case when it covers both of these vertices. The number of maximal matchings covering them with the edge uw is the number of maximal matchings in $B(n,k)\backslash\{u,w\}$. This graph is a disjoint union of n paths P_k, each of them has ψ_k maximal matchings. Therefore, the number of maximal matchings covering vertices u and w by the same edge is equal to

$$\psi_k^n. \tag{2.3}$$

Let us now consider the case when u and w are covered by different edges in a maximal matching. If both of these edges are in the same cycle, say uv_{l1} and wv_{lk}, what remains when we remove them, is a disjoint union of $n-1$ copies of P_k and one copy of P_{k-2}. The number of maximal matchings in that case is $\psi_k^{n-1}\psi_{k-2}$. Since there are n possibilities for choosing the cycle in which the edges covering u and w are, the total number of such maximal matchings is

$$n\,\psi_{k-2}\,\psi_k^{n-1}. \tag{2.4}$$

In the same way, we can conclude that the number of maximal matchings in which u and w are covered by edges from different cycles is equal to

$$n(n-1)\,\psi_{k-1}^2\,\psi_k^{n-2}. \tag{2.5}$$

The last possible case is that the maximal matching covers only one of the vertices u and w, say u. Let it be covered by the edge uv_l. Then all the neighboring vertices of w must be covered by edges $v_{1k}v_{1,k-1}, ..., v_{nk}v_{n,k-1}$. So, we have one copy of P_{k-3} and $n-1$ copies of P_{k-2}, with n such possible situations, so we have $n\psi_{k-3}\psi_{k-2}^{n-1}$ maximal matchings that cover only u. By symmetry, there are exactly as many maximal matchings that cover only w, so the number of maximal matchings that cover only one of the vertices u and w is equal to

$$2n\,\psi_{k-3}\,\psi_{k-2}^{n-1}. \tag{2.6}$$

Now we get the total number of maximal matchings in $B(n,k)$ for $n \geq 2, k \geq 3$ by summing all possible cases.

□

For the same reason, as in the proof for maximal matchings, we state the result for saturation number, where n denotes the number of vertices [2],

$$s(P_n) = \left\lfloor \frac{n+1}{3} \right\rfloor \tag{2.7}$$

and matching number for paths

$$\nu(G) = \left\lfloor \frac{n}{2} \right\rfloor. \tag{2.8}$$

Proposition 2.4. *Saturation number for graph $B(n,k)$ is equal to*

$$s(B(n,k)) = \begin{cases} (n-1)\left\lfloor \frac{k-1}{3} \right\rfloor + \left\lfloor \frac{k-2}{3} \right\rfloor, & \text{if } k \geq 3 \\ 2\left\lfloor \frac{k}{3} \right\rfloor + (n-2)\left\lfloor \frac{k+1}{3} \right\rfloor, & \text{if } k = 2. \end{cases}$$

Proof. Let's consider all possible cases. Any maximal matching in $B(n,k)$ must cover at least one of the vertices u and w.
We first look at the case when it covers both of these vertices. This graph is a disjoint union of n paths P_k, and each of them has saturation number $s(P_k) = \left\lfloor \frac{k+1}{3} \right\rfloor$. Therefore, saturation number in this case is equal to

$$n\left\lfloor \frac{k+1}{3} \right\rfloor + 1 \tag{2.9}$$

In the case when u and w are covered by different edges we have two options. If both of these edges are in the same cycle, for the disjoint union of $n-1$ copies of P_k and one copy of P_{k-2} saturation number is equal

$$(n-1)\left\lfloor\frac{k+1}{3}\right\rfloor+\left\lfloor\frac{k-1}{3}\right\rfloor \tag{2.10}$$

In the same way, we can conclude that the saturation number in case when u and w are covered by edges from different cycles is equal to

$$2\left\lfloor\frac{k}{3}\right\rfloor+(n-2)\left\lfloor\frac{k+1}{3}\right\rfloor \tag{2.11}$$

If the maximal matching covers only one of the vertices u and w, then we have one copy of P_{k-3} and $n-1$ copies of P_{k-2}, so the saturation number is equal to

$$(n-1)\left\lfloor\frac{k-1}{3}\right\rfloor+\left\lfloor\frac{k-2}{3}\right\rfloor, \tag{2.12}$$

which is also the smallest possible value for saturation number. For $k=2$ formulas (11) and (12) give the same value.

□

By analogical consideration it can be easily shown that the formula for matching number in $B(n,k)$ is given by the following proposition.

Proposition 2.5.

$$\nu(B(n,k))=n\left\lfloor\frac{k}{2}\right\rfloor+1.$$

Proof. In the first case when maximal matching covers both of joint vertices, and graph is a disjoint union of n paths P_k, is also maximum matching in $B(n,k)$.

□

3 Concluding remarks

In this paper we have enumerated maximal matchings in a class of cycle related graphs, interesting from the viewpoint of topology of computer networks. Our results could be generalized in a straightforward way to similar network configurations, in particular to multiple ring networks sharing a single node, and we leave it to the interested reader. For both types of networks, several interesting problems remain unanswered. Of particular interest would be to compare the results for two cycles with a larger number of vertices compared to cases with multiple cycles with a smaller number of vertices. Some preliminary investigations are underway and we hope to be able to report conclusive results soon.

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