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Divisions of narrow strips in square and hexagonal lattices

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Abstract

In a recent paper, it was determined that the total number of divisions of a narrow hexagonal strip is counted by odd-indexed Fibonacci numbers. In this paper, we consider two division problems on narrow strips of square and hexagonal lattices. In both cases, we compute the bivariate enumerating sequences and the corresponding generating functions, which allowed us to determine the asymptotic behavior of the total number of such subdivisions and the expected number of parts. For the square case, we extend the results of two recent references by establishing the polynomiality of enumerating sequences forming columns and diagonals of the triangular enumerating sequence. In the hexagonal case, we provide an alternative proof for the number of divisions. We also show how both cases could be treated via the transfer matrix method and discuss some directions for future research.

1 Introduction

It has been a long-standing problem of great practical importance to count the ways of dividing a collection of entities into smaller sets according to a given set of rules. If the entities are considered to be indivisible, and we only care about their number, the natural framework for modeling such situations is the theory of integer partitions and compositions, depending on further properties of the considered entities. If, on the other hand, we are interested in relationships between the entities, such as e.g., their adjacency

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patterns or their relative positions, we must resort to more complex models such as graphs and geometric figures.
In this paper, we look at finite portions of the square and hexagonal regular lattices and count ways of dividing narrow strips in such lattices into a given number of pieces while preserving the integrity of individual squares or hexagons. The considered portions of square and hexagonal lattices remind us of chocolate bars and honeycomb slabs, respectively, hence the title. We start by revisiting some partial results for narrow strips in the square lattice available in the literature and present a complete solution to the problem. In particular, we derive the recurrences satisfied by the sequences enumerating the divisions of a $2 \times n$ strip into k pieces. From them, we compute the bivariate generating function whose univariate specialization yields the recurrence for the overall number of divisions. In that way, we recover the results of Knopfmacher obtained in the context of compositions of ladder graphs [5]. We refine those results by investigating the behavior of columns in the enumerating triangle. We establish convolution-type recurrences for all columns, going thus beyond partial results of references [2, 4, 7]. Then we apply the same approach to narrow strips of hexagons, again deriving the recurrences and computing the bivariate generating function. Then we show how the results for honeycomb strips can be obtained by using transfer matrices. Finally, we also derive transfer matrices for the chocolate bars from which we started.
The paper is concluded with some remarks on the strong and weak points of employed methods and with some indications of possible further directions.

2 Definitions and preliminary results

Let n be a non-negative integer. We consider a $2 \times n$ rectangular strip consisting of $2n$ squares arranged in 2 rows and n columns, such as the one shown in Fig. 1. In the rest of the paper, we will often refer to such strips as chocolate bars of length n. We consider divisions of such structures into a given number of pieces obtained by cutting along the edges of basic squares. More precisely, we would like to find the number of all possible divisions of such a bar of a given length, and also the number of such divisions into a given number of parts k. Clearly, $1 \leq k \leq 2n$ are the only meaningful values of k. Let $r_k(n)$ denote the number of divisions of $2 \times n$ rectangular strip into exactly k pieces and $r(n)$ the total number of divisions. From definition we have that $r_k(n) = 0$ for $k < 1$ and for $k > 2n$. The initial values are $r_1(1) = r_2(1) = 1$, and $r_1(2) = 1$, $r_2(2) = 6$, $r_3(2) = 4$ and $r_1(4) = 1$.
In a recent paper, Brown [2] studied such divisions and obtained a system of recursive relations that we include below as Theorem 2.1. In order to state

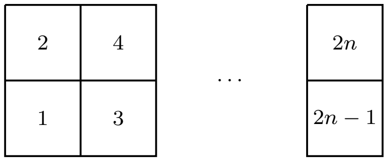

Figure 1: Rectangular strip containing $2n$ squares.

Brown's results, we need one auxiliary term, more specifically, the number of divisions of a $2 \times n$ rectangular strip into k parts such that the squares in the last column belong to different parts. We denote that number by $q_k(n)$ and show one such division in Figure 2 as an example.

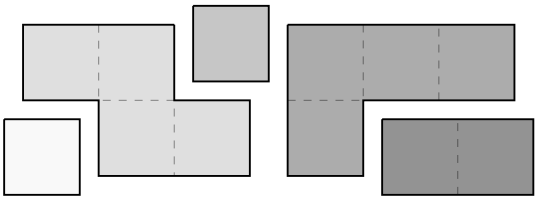

Figure 2: One division of 2×6 rectangular strip into 5 parts with squares in the last column being in the different parts. The total number of such divisions is denoted by $q_5(6)$.

Theorem 2.1 (Brown)**.** *The number of divisions of* $2 \times n$ *strip into* k *parts satisfies following system of equations:*

$$\begin{aligned} r_k(n+1) &= r_k(n) + 3r_{k-1}(n) + r_{k-2}(n) + 2q_k(n) \\ q_k(n+1) &= 2r_{k-1}(n) + r_{k-2}(n) + q_k(n). \end{aligned}$$

It is an easy exercise to eliminate $q_k(n)$ from the system of equations in Theorem 2.1 and to obtain recursive relations for $r_k(n)$,

$$r_k(n+1) = r_{k-2}(n) + 3r_{k-1}(n) + 2r_k(n) + r_{k-2}(n-1) + r_{k-1}(n-1) - r_k(n-1), \tag{2.1}$$

and for the overall number of such divisions,

$$r(n+1) = 6r(n) + r(n-1). \tag{2.2}$$

These recurrences will serve as the starting point of our Section 3, where Brown's results will be extended and refined by establishing recurrences in n for a fixed k and by computing the expected values of k in a random division of a $2 \times n$ chocolate bar.

Next, we consider a hexagonal strip composed of n regular hexagons as shown in Figure 3. Throughout the paper, such hexagonal strips will also be referred to as honeycomb strips. The number to divide hexagonal strips with n hexagons into exactly k parts is given by $\binom{n+k-2}{n-k}$ while the overall number of divisions is F_{2n-1}, where F_n stands for the nth Fibonacci number [3]. Here we present recurrences, generating functions, and again, only divisions along the edges of hexagons are considered. The hexagons are added in the order as it is shown in Figure 3.

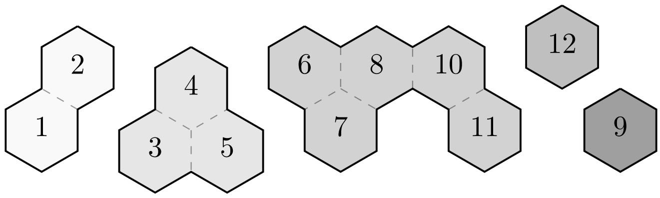

Figure 3: Honeycomb strip with 12 hexagons divided into 5 pieces.

Let $D_k(n)$ denote the set of all possible divisions of the honeycomb strip with n hexagons into k pieces and $d_k(n) = |D_k(n)|$ the number of elements of the set $D_k(n)$. Now we can state some simple cases: $d_1(n) = 1$, for every non-negative integer n, since there is only one way to obtain one part, and $d_n(n) = 1$, since there is only one way to obtain n parts, that is to let each hexagon form its own part. Furthermore, $d_k(n) = 0$ for $k < 1$ and for $k > n$. It is convenient to set $d_1(0) = 1$. As an example, we list all possible divisions of the strip containing 4 hexagons as the first non-trivial case.

$$\begin{aligned}
d_1(4) &= 1 \quad \{1234\} \\
d_2(4) &= 6 \quad \{1,234\},\{2,134\},\{3,124\},\{4,123\},\{12,34\},\{13,24\} \\
d_3(4) &= 5 \quad \{12,3,4\},\{13,2,4\},\{23,1,4\},\{24,1,3\},\{34,1,2\} \\
d_4(4) &= 1 \quad \{1,2,3,4\}
\end{aligned}$$

Note that the division $\{14, 23\}$ is not included, since hexagons 1 and 4 are not adjacent, as shown in Figure 4, thus cannot form a part.
Since the inner dual of a $2 \times n$ rectangular strip is a subgraph of the inner dual of a hexagonal strip of length $2n$, all divisions of a $2 \times n$ rectangular strip are also valid divisions of a hexagonal strip with $2n$ hexagons, but not vice versa. Figure 5 shows the division $\{1, 23, 4\}$ which is legal in the hexagonal strip but illegal in the rectangular strip.

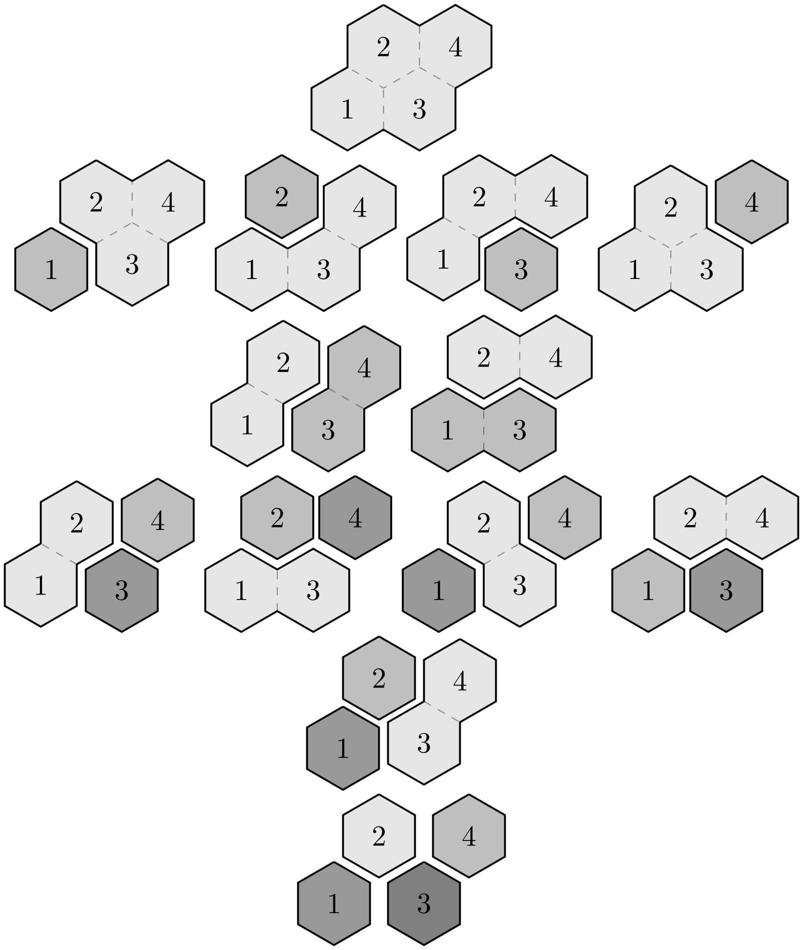

Figure 4: Divisions of the strip with 4 hexagons.

3 Dividing a chocolate bar into a given number of parts

Numbers $r_k(n)$ of (2.1) form a triangular array; its first few lines are shown in Table 1. In this section, we investigate the behavior of its columns, i.e., we turn our attention to recursive relation for $r_k(n)$ where k is fixed.
As mentioned before, $r_0(n) = 0$ and $r_1(n) = 1$, so we look at the first non-trivial case, $k = 2$. From relation (2.1) we obtain $r_2(n) = r_0(n-1) + 3r_1(n-1) + 2r_2(n-1) + r_0(n-2) + r_1(n-2) - r_2(n-2)$. By plugging in

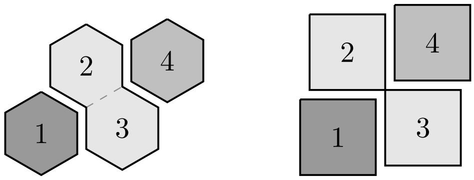

Figure 5: A valid division of a honeycomb strip on the left, and the corresponding division of a rectangular grid that is not allowed on the right side.

$n\backslash k$	1	2	3	4	5	6	7	8	9	10
1	1	1								
2	1	6	4	1						
3	1	15	29	21	7	1				
4	1	28	107	153	111	45	10	1		
5	1	45	286	678	831	603	274	78	13	1

Table 1: First few rows of $r_k(n)$.

$r_0(n-1) = r_0(n-2) = 0$ and $r_1(n-1) = r_1(n-2) = 0$, we obtain

$$\begin{aligned} r_2(n) =& 2r_2(n-1) - r_2(n-2) + 4 \\ r_2(n-1) =& 2r_2(n-2) - r_2(n-3) + 4, \end{aligned}$$

and by subtracting these two equations, we arrive at

$$r_2(n) = 3r_2(n-1) - 3r_2(n-2) + r_2(n-3). \tag{3.1}$$

Rewriting the trivial case

$$r_1(n) = r_1(n-1) \tag{3.2}$$

as

$$\binom{1}{0} r_1(n) = \binom{1}{1} r_2(n-1), \tag{3.3}$$

and case $k = 2$ as

$$\binom{3}{0} r_2(n) = \binom{3}{1} r_2(n) - \binom{3}{2} r_2(n-1) + \binom{3}{3} r_2(n-2) \tag{3.4}$$

suggests that there is a pattern valid also for higher values of k. The conjectured pattern is readily verified by induction, thus yielding the following theorem.

Theorem 3.1. *For integers $n, k \geq 1$ we have*

$$\sum_{j=0}^{2k-1} (-1)^j \binom{2k-1}{j} r_k(n-j) = 0. \tag{3.5}$$

Proof. The proof is by induction. For $k = 1, 2$, the base of induction is true, as stated above. To verify the step of induction, we use recursion (2.1) to obtain a system of $2k-2$ equations as follows:

$$\begin{aligned}
r_k(n) = & r_{k-2}(n-1) + 3r_{k-1}(n-1) + 2r_k(n-1) + \\
& + r_{k-2}(n-2) + r_{k-1}(n-2) - r_k(n-2) \\
r_k(n-1) = & r_{k-2}(n-2) + 3r_{k-1}(n-2) + 2r_k(n-2) + \\
& + r_{k-2}(n-3) + r_{k-1}(n-3) - r_k(n-3) \\
r_k(n-2) = & r_{k-2}(n-3) + 3r_{k-1}(n-3) + 2r_k(n-3) + \\
& + r_{k-2}(n-4) + r_{k-1}(n-4) - r_k(n-4) \\
& \vdots \\
r_k(n-2k+3) = & r_{k-2}(n-2k+2) + 3r_{k-1}(n-2k+2) + 2r_k(n-2k+2) + \\
& + r_{k-2}(n-2k+1) + r_{k-1}(n-2k+1) - r_k(n-2k+1)
\end{aligned}$$

The term $r_k(n-j)$ appears in at most three equations, namely in the $(j-1)^{\text{st}}$, j^{th} and $(j+1)^{\text{st}}$ equation. To proceed forward, we multiply j-th equation by $(-1)^j \binom{2k-3}{j-1}$ and we add up all equations. For even j, the term $r_k(n-j)$ appears with the coefficient

$$\binom{2k-3}{j-2} + 2\binom{2k-3}{j-1} + \binom{2k-3}{j} = \binom{2k-1}{j},$$

and for odd j with the same coefficient, but with the opposite sign. We conclude that

$$r_k(n) = \sum_{j=1}^{2k-1} (-1)^j \binom{2k-1}{j} r_k(n-j) + A_{k-1}(n) + A_{k-2}(n), \tag{3.6}$$

where $A_{k-1}(n)$ and $A_{k-2}(n)$ are some expressions involving $r_{k-1}(n-j)$ and $r_{k-2}(n-j)$, respectively. The claim of the theorem will be established if we show that both $A_{k-1}(n)$ and $A_{k-2}(n)$ are equal to zero. We first look at $A_{k-1}(n)$. For $j \geq 1$, the term $r_{k-1}(n-j)$ appears twice in our system of equations, in the $(j-1)^{\text{st}}$ and in the j^{th} equation, hence, the coefficient by $r_{k-1}(n-j)$ is $\binom{2k-3}{j-2} - 3\binom{2k-3}{j-1}$ for an odd j, and $3\binom{2k-3}{j-1} - \binom{2k-3}{j-2}$ for an even j. So,

$$A_{k-1}(n) = \sum_{j=1}^{2k-1} (-1)^j \left(3\binom{2k-3}{j-1} - \binom{2k-3}{j-2} \right) r_{k-1}(n-j).$$

For $k-1$ we can use the induction hypothesis, hence

$$3\sum_{j=0}^{2k-3}(-1)^j\binom{2k-3}{j}r_{k-1}(n-j-1)=0$$

and

$$\sum_{j=0}^{2k-3}(-1)^j\binom{2k-3}{j}r_{k-1}(n-j-2)=0.$$

After adding the equations, we have

$$\begin{aligned}0&=3\sum_{j=0}^{2k-3}(-1)^j\binom{2k-3}{j}r_k(n-j-1)+\sum_{j=0}^{2k-3}(-1)^j\binom{2k-3}{j}r_k(n-j-2)\\&=3\sum_{j=1}^{2k-2}(-1)^{j-1}\binom{2k-3}{j-1}r_k(n-j)+\sum_{j=2}^{2k-1}(-1)^j\binom{2k-3}{j-2}r_k(n-j)\\&=\sum_{j=1}^{2k-1}(-1)^j\left(\binom{2k-3}{j-2}-3\binom{2k-3}{j-1}\right)r_k(n-j),\end{aligned}$$

hence, $A_{k-1}(n)=0$.
Similarly, $A_{k-2}(n)$ can be expressed as

$$A_{k-2}(n)=\sum_{j=1}^{2k-1}(-1)^j\left(\binom{2k-3}{j-1}-\binom{2k-3}{j-2}\right)r_{k-2}(n-j),$$

and, again, by using the induction hypothesis for $k-2$, we obtain $A_{k-2}(n)=0$. The proof follows along the same lines as for $A_{k-1}(n)=0$, and we omit the details.
This completes our proof. □

Theorem 3.1 implies that all columns of the array $r_k(n)$ are polynomials in n. Moreover, $r_k(n)$ is a polynomial in n of degree $2k-2$. The exact expressions can be easily obtained by fitting to the initial values, but we omit the details. Our Theorem 3.1 reestablishes the polynomiality results of references [2] and [4] in a more compact and self-contained form.
A similar reasoning could also be employed near the upper end of the range of k and used to establish polynomiality of diagonals $r_{2n-k}(n)$, going thus beyond the results of references [2, 4]. Indeed, $r_{2n}(n)=1$ for all non-negative integers n. Furthermore, $r_{2n-1}(n)=3n-2$, since among the $2n-1$ pieces there must be exactly one dimer. That dimer is an edge in the inner dual of our bar, hence an edge in a ladder graph with n rungs, and there are exactly $3n-2$ such edges. In a similar way, one can see that

$r_{2n-2}(n)$ must be a quadratic polynomial in n: A division into $2n-2$ parts can either contain one trimer and $2n-3$ monomers, or two dimers and $2n-4$ monomers. As the number of trimers is linear in n and the number of pairs of dimers is quadratic in $3n$, by fitting on the first few values for small n one obtains $r_{2n-2}(n) = \frac{9}{2}(n-1)(n-\frac{2}{3})$. By continuing with the same reasoning, one obtains a general result.

Theorem 3.2. *$r_{2n-k}(n)$ is a polynomial of degree k in n with the leading coefficient $\frac{3^k}{k!}$.*

We leave the details to the interested reader.
We now move towards computing the bivariate generating function for $r_k(n)$. Let

$$F(x,y) = \sum_{n\geq 1}\sum_{k\geq 1} r_k(n)x^n y^k$$

denote the desired generating function. By starting from recurrence (2.1) we readily obtain

$$F(x,y) = \frac{xy(1-x+y+xy)}{1-(2+3y+y^2)x-(y^2+y-1)x^2}.$$

By substituting $y=1$ we obtain

$$F(x,1) = \frac{2x}{1-6x-x^2},$$

the univariate generating function for the sequence r_n.
Now we can determine the expected number of pieces in a random division. We rely on the following version of Darboux's theorem [1].

Theorem 3.3 (Darboux)**.** *If the generating function $f(x) = \sum_{n\geq 0} a_x^n$ of a sequence (a_n) can we written in the form $f(x) = \left(1-\frac{x}{\omega}\right)^{\alpha} h(x)$, where ω is the smallest modulus singularity of f and h is analytic in ω, then $a_n \sim \frac{h(\omega)n^{-\alpha-1}}{\Gamma(-\alpha)\omega^n}$, where Γ denotes the gamma function.*

Since $\omega = \sqrt{10}-3$ we can write

$$F(x,1) = \frac{2x}{x\left(\sqrt{10}-3\right)+1}\left(1-\frac{x}{\left(\sqrt{10}-3\right)}\right)^{-1}.$$

Hence, we have $h(x) = \dfrac{2x}{x\left(\sqrt{10}-3\right)+1}$ and $h(\omega) = \frac{\sqrt{10}}{10}$. Furthermore,

$$\left.\frac{\partial F(x,y)}{\partial y}\right|_{y=1} = \frac{-x(x^3+3x^2+7x-3)}{\left(x\left(\sqrt{10}-3\right)x+1\right)^2}\left(1-\frac{x}{\left(\sqrt{10}-3\right)}\right)^{-2}$$

yields $g(x) = \frac{-x(x^3+3x^2+7x-3)}{\left(x(\sqrt{10}-3)x+1\right)^2}$ and $g(\omega) = \frac{3\sqrt{10}-4}{20}$. By Theorem 3.3, the expected number of parts is

$$\frac{\frac{g(\omega)n}{\Gamma(2)\omega^n}}{\frac{h(\omega)}{\Gamma(1)\omega^n}} = \left(\frac{3}{2} - \sqrt{\frac{2}{5}}\right) n.$$

Hence, we have established the following result for the expected number of parts in a random division of a chocolate bar of length n.

Theorem 3.4. *The expected number of parts in a random division of a chocolate bar of length n is given by*

$$\left(\frac{3}{2} - \sqrt{\frac{2}{5}}\right) n \approx 0.867544\, n.$$

The above result is derived under the so-called equilibrium assumption, where all divisions are equally likely.
The triangle of Table 1 is not (yet) in the OEIS [6]. However, its row sums appear as A078469, the number of compositions of ladder graphs in the sense of reference [5]. Hence, our results could also be interpreted as a refinement of the number of compositions of ladder graphs. Sequence $r_3(n)$ appears as A345897, with the same interpretation as we give here. Curiously, such an interpretation seems to be missing among many combinatorial interpretations of A000384, the hexagonal numbers, which appear as the second column of our triangle. Similarly, $r_{2n-2}(n)$ appears as A081266, but without the interpretation given here.

4 Divisions of honeycomb strips

4.1 Recurrences, explicit formulas and generating functions

Recall that $D_k(n)$ denotes the set of all possible divisions of the honeycomb strip with n hexagons into k pieces, and $d_k(n) = |D_k(n)|$ is the number of elements of the set $D_k(n)$. In order to count the divisions correctly, special attention must be paid to the rightmost two cells, since the new $(n+1)^{\text{st}}$ cell can interact only with them. Whether these hexagons are in the same pieces or not plays a crucial role in how the new hexagon can be added. We denote by $S_k(n)$ the set of all possible divisions of the honeycomb strip with n hexagons into k pieces, with the last two hexagons in the different parts. Similarly, let $T_k(n)$ denote the set of all possible divisions of the strip into k pieces with the two rightmost hexagons belonging to the same piece. Let

$s_k(n) = |S_k(n)|$ and $t_k(n) = |T_k(n)|$. Since the last two hexagons can either be together or separated, we have divided the set $D_k(n)$ into two disjoint sets, $D_k(n) = S_k(n) \cup T_k(n)$, hence $d_k(n) = t_k(n) + s_k(n)$.

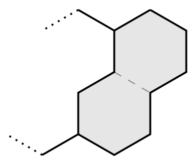

Figure 6: A honeycomb strip with two rightmost hexagons in the same piece.

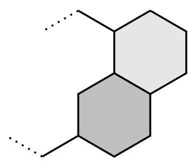

Figure 7: A honeycomb strip with two rightmost hexagons in different pieces.

We first establish an auxiliary result.

Theorem 4.1. *For $n \geq 1$, the number of all possible divisions $s_k(n)$ of the honeycomb strip with n hexagons into k pieces, with hexagons in the last column being in the different pieces, satisfies the following relation:*

$$s_k(n+1) = s_{k-1}(n) + 2s_k(n) - s_k(n-1). \tag{4.1}$$

Proof. We start with a strip containing n hexagons and add one new hexagon to obtain a strip with $n+1$ hexagons. The new hexagon can either increase the number of parts in the division by 1 or not increase this number. To obtain a division with k pieces, we can only start with the division with $k-1$ or k pieces. These are two disjoint sets, so the number of all divisions will be the sum of these cases.
When starting with division consisting of $k-1$ pieces, we can obtain k pieces by adding the new hexagons as individual pieces. Since there is only one way to do that, the number of divisions that can be obtained this way is $d_{k-1}(n)$. Note that the condition that the rightmost two hexagons belong to different pieces is satisfied, as shown in Figure 8.
It remains to consider one last case. We start with a strip divided into k pieces and we add $(n+1)^{\text{st}}$ hexagon. If the last two hexagons in the division are together, we cannot add new hexagons so that the number of parts remains the same, and the new hexagons are in different pieces.

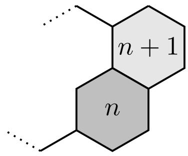

Figure 8: The element of $S_k(n+1)$ obtained from the element of $D_{k-1}(n)$ by adding the new hexagon as separated piece.

Now we move to the case where the last two hexagons in the division are separated. There is only one way to add new hexagons to the existing strip, to put the $(n+1)^{\text{st}}$ hexagon together with $(n-1)^{\text{st}}$ (see Figure 9). Every other layout would be in contradiction with either the number of pieces or the fact that the last two hexagons should be separated, since putting $(n+1)^{\text{st}}$ hexagon together with n^{th} hexagon would produce the element of $T_k(n)$. So in this case, we have $s_k(n)$ ways to obtain the desired division.

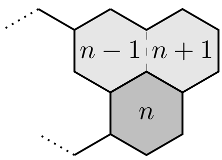

Figure 9: The element of $S_k(n+1)$ obtained from the element of $S_k(n)$ by joining the new hexagon with $(n-1)^{\text{st}}$ hexagon

By summing these two cases, we obtain the recursive relation

$$s_k(n+1) = d_{k-1}(n) + s_k(n). \tag{4.2}$$

To eliminate $d_{k-1}(n)$ from relation (4.2), we use the fact that $d_{k-1}(n) = t_{k-1}(n)+s_{k-1}(n)$. By removing the last hexagon from the strip, we establish a 1-to-1 correspondence between all divisions of a strip with $n-1$ hexagons and divisions of a strip with n hexagons where the last two hexagons are in the same part. Hence, $t_k(n) = d_k(n-1)$. Then we have

$$\begin{aligned} s_k(n+1) &= s_{k-1}(n) + t_{k-1}(n) + s_k(n) \\ &= s_{k-1}(n) + d_{k-1}(n-1) + s_k(n), \end{aligned}$$

hence $d_{k-1}(n-1) = s_k(n+1) - s_{k-1}(n) - s_k(n)$, which combined with relation (4.2) yields

$$s_k(n+1) = s_{k-1}(n) + 2s_k(n) - s_k(n-1)$$

and we proved the theorem. □

By disregarding values of k in recursive relation 4.1 we obtain

$$s(n+1) = 3s(n) - s(n-1),$$

where $s(n)$ represents the number of all divisions of a honeycomb strip of length n with the last two hexagons in different parts. Since we obtained the same recursive relation as for bisection of Fibonacci sequence with $s(1) = 0$ and $s(2) = 1$, we have

$$s(n) = F_{2n-2}.$$

Our main result of this section now follows by much the same reasoning as $d_k(n)$ satisfies the same recurrence as $s_k(n)$. We state it without proof.

Theorem 4.2. *For $n \geq 1$, the number of all possible divisions $d_k(n)$ of n honeycomb strip into k pieces satisfies the following relation:*

$$d_k(n+1) = d_{k-1}(n) + 2d_k(n) - d_k(n-1). \tag{4.3}$$

Again, by grouping terms of recurrence 4.3 with respect to n, we obtain the recurrence satisfied by the sequence $d(n)$ counting the total number of subdivisions of a honeycomb strip of length n as

$$d(n+1) = 3d(n) - d(n-1).$$

Taking into account the initial conditions $d(1) = 1$ and $d(2) = 2$ yields a very simple answer.

Theorem 4.3. *The total number of divisions of a honeycomb strip of length n is given by $d(n) = F_{2n-1}$, where F_n denotes the n^{th} Fibonacci number.*

The above theorem yields a nice combinatorial interpretation of the odd-indexed Fibonacci numbers.

With the above result at hand, it is not too difficult to guess the explicit formulas for $d_k(n)$ and $s_k(n)$. The following theorem is easily proved by simply verifying that the proposed expressions satisfy the respective recurrences and initial conditions.

Theorem 4.4. *The number of all divisions $d_k(n)$ of the honeycomb strip with n hexagons into exactly k pieces is*

$$d_k(n) = \binom{n+k-2}{n-k}.$$

The number $s_k(n)$ of all divisions of the honeycomb strip with n hexagons into k pieces such that the two rightmost hexagons belong to different pieces is equal to zero if $n = 1$ and for $n \geq 2$ it is given as

$$s_k(n) = \binom{n+k-3}{n-k}.$$

Even though sequences $r_k(n)$ and $d_k(n)$ satisfy different recursive relations and describe different problems, it turns out that their columns satisfy the same recurrences. Our next theorem is analogous to Theorem 3.1, but for the sequence $d_k(n)$. We state it without proof.

Theorem 4.5. *For $n, k \geq 1$ we have*

$$\sum_{j=0}^{2k-1} (-1)^j \binom{2k-1}{j} d_k(n-j) = 0. \tag{4.4}$$

As with a rectangular strip, we are now interested in the generating function of the sequence $d_k(n)$. Let

$$G(x,y) = \sum_{n\geq 1} \sum_{k\geq 1} d_k(n) x^n y^k.$$

By recursive relation 4.3 we have

$$\begin{aligned} G(x,y) &= xy + x^2 y\,(1+y) + \\ &\quad + \sum_{n\geq 3} \sum_{k\geq 1} \left(d_{k-1}(n-1) + 2d_k(n-1) - d_k(n-2)\right) x^n y^k \\ &= xy + x^2 y\,(1+y) + xy\,(G(x,y) - xy) + \\ &\quad + 2x\,(G(x,y) - xy) - x^2 G(x,y), \end{aligned}$$

so we have

$$G(x,y) = \frac{xy(1 + x(y-1) - xy)}{1 - (2+y)x + x^2}.$$

By putting $y = 1$, we obtain the univariate generating function for the sequence $d(n)$ as

$$G(x,1) = \frac{x - x^2}{1 - 3x + x^2}.$$

Its smallest-modulus singularity is $\omega = \frac{1}{2}\left(3 - \sqrt{5}\right)$ and this gives us the asymptotics of the expected number of pieces in a random divisions of honeycomb strips of a given length.

Theorem 4.6. *The expected number of pieces in a random division of a honeycomb strip of length n asymptotically behaves as*

$$\frac{\sqrt{5}}{5} n \approx 0.447214n.$$

The proof follows by a straightforward application of the Darboux theorem, and we omit the details.

4.2 Some consequences

Our results make it possible to give a new combinatorial interpretation for some famous identities. We present two such cases.
First, by double-counting the set $D(n)$, we gave new meaning to the well-known identity

$$\sum_{k=1}^{n} \binom{n+k-2}{n-k} = F_{2n-1}.$$

Another identity will be proven in the next theorem.

Theorem 4.7. *For $n, m \geq 1$ we have*

$$F_{2n+2m-1} = F_{2n-1}F_{2m-1} + F_{2n}F_{2m}.$$

Proof. We start with two honeycomb strips of lengths n and m. To prove the statement of a theorem, we glue strips together as in Figure 10 and double-count the number of divisions. On one hand, we have a strip of length $n+m$ whose number of divisions is $d(n+m)$. On the other hand, we consider what can happen when strips are glued together. In the first case, parts of each division do not interact, hence, we have $d(n)d(m)$ such divisions. In the other cases, at least two parts, one from each strip, must merge. But to correctly count the number of divisions in those cases, it is important to know whether the division is with the two last hexagons together or separated. If both strips have the last two hexagons together, the total number of such divisions is $t(n)t(m)$. If both strips have the last two hexagons separated, the total number of such divisions is $4s(n)s(m)$, since there are four different ways to merge parts. Finally, if one strip has two last hexagons separated and the other one together, we can merge the parts in two ways. Since either one of the strips can be in both situations, the total number of divisions in this case is $2s(n)t(m) + 2s(n)t(m)$.

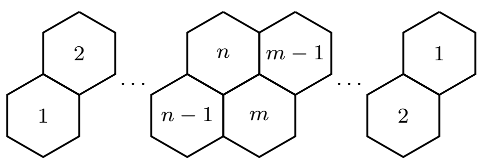

Figure 10: Two honeycomb strips glued together.

So,

$$d(n+m) = d(n)d(m) + 4s(n)s(m) + t(n)t(m) + 4s(n)t(m).$$

The claim now follows by substituting $d(n) = F_{2n-1}$, $s(n) = F_{2n-2}$ and $t(n) = F_{2n-3}$ and rearranging the resulting expressions. $\square$

The results of this section can also be formulated in terms of graph compositions, this time of the graph P_n^2 obtained by adding edges between all pairs of vertices at distance 2 in P_n, the path on n vertices. The following results are a direct consequence of the fact that P_n^2 is the inner dual of a honeycomb strip of length n.

Theorem 4.8. *The number of compositions of P_n^2 with k components is equal to $\binom{n+k-2}{n-2}$. The total number of compositions of P_n^2 is equal to F_{2n-1}.*

5 Transfer matrix method

5.1 Honeycomb strips

In this section, we present another approach to obtain an overall number of divisions, the one based on transfer matrices. It might seem less natural than recurrence relations, but it often turns out to be suitable when recurrence relations are complicated or unknown.

We again consider a honeycomb strip such as the one shown in the Figure 3, and look at its rightmost column, i.e., at the hexagons labeled by $n-1$ and n. There are two possible situations regarding these hexagons: they can be in the same piece of a subdivision, or they can belong to two different pieces. We denote a strip with the last two hexagons together as a type T strip and a strip with the last two hexagons separated as a type S strip. Adding the $(n+1)-$st hexagon might result again in a type S strip or a type T strip. There are altogether four possibilities, each of which produces certain effects on the number of pieces in the resulting strip. For example, if we start with a strip of type S and we want to end with a strip of type S, we can either add the new hexagon to the part which contains the $(n-1)^{\text{st}}$ hexagon, or we can let the $(n+1)^{\text{st}}$ hexagon to form its own part. In the first case, the number of parts will remain the same; in the second case, it will increase by one. Figure 11 shows this case. The main idea of the transfer matrix

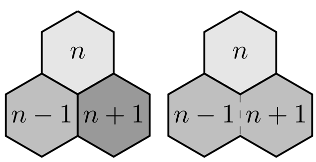

Figure 11: Both cases resulting in a strip of type S.

method is to arrange the effects of adding a single hexagon into a 2×2 matrix whose entries will keep track of the number of pieces via a formal variable, say, y. The rows and columns of such a matrix are indexed by possible states, in our case T and S, and the element at the position S, S in our example will be $1+y$. That clearly captures the fact that transfer from S to S results either in the same number of pieces, or the number of pieces increases by one. The other three possible transitions, $T \to T$, $S \to T$ and $T \to S$ are described by matrix elements $1, 1$, and y, respectively. Indeed, it is clear that adding a hexagon to obtain the rightmost column together cannot increase the number of pieces, hence the two ones, and that starting from T and arriving at S is possible only by the last hexagon forming a new piece, hence increasing the number of pieces by one, hence y. If we denote our matrix by H, we can write it as

$$H(y) = \begin{bmatrix} 1 & 1 \\ y & 1+y \end{bmatrix}.$$

By construction, it is clear that adding a new hexagon will be well described by multiplying some vector of states by our matrix $H(y)$, and that repeated addition of hexagons will correspond to multiplication by powers of $H(y)$. It remains to account for the initial conditions.
For the initial value $n = 1$, we have a trivial case, one hexagon forms one part. For $n = 2$ we have two possibilities, hexagons are in the same part or separated. Hence, his case is represented by a vector

$$\overrightarrow{h_2} = x \begin{bmatrix} y \\ y^2 \end{bmatrix}.$$

By introducing another formal variable, say x, to keep track of the length, the above procedure will produce a sequence of bivariate polynomials whose coefficients are our numbers $d_k(n)$. The first few polynomials are shown in Table 2 after the theorem, which summarizes the described procedure.

Theorem 5.1. *The number of divisions of a honeycomb strip of a length n into k parts is the coefficient of $x^n y^k$ in the expression*

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ y & 1+y \end{bmatrix}^{n-2} \begin{bmatrix} y \\ y^2 \end{bmatrix} x^n. \tag{5.1}$$

The coefficients by $x^n y^k$ in expression (5.1) could now be determined by studying the powers of the transfer matrix. By looking at the first few cases,

$$H(y)^2 = \begin{bmatrix} 1+y & 2+y \\ 2y+y^2 & 1+3y+y^2 \end{bmatrix}$$

and

$$H(y)^3 = \begin{bmatrix} 1+3y+y^2 & 3+4y+y^2 \\ 3y+4y^2+y^3 & 1+6y+5y^2+y^3 \end{bmatrix},$$

n	
1	xy
2	$x^2(y+y^2)$
3	$x^3(y+3y^2+y^3)$
4	$x^4(y+6y^2+5y^3+y^4)$
5	$x^5(y+10y^2+15y^3+7y^4+y^5)$
6	$x^6(y+15y^2+35y^3+28y^4+9y^5+y^6)$

Table 2: First few bivariate polynomials from the transfer matrix method.

we could guess the entries in the general case and then verify them by induction. We state the result, omitting the details of the proof.

Lemma 5.2. *Matrix*

$$H(y)^n = \begin{bmatrix} p(n) & s(n) \\ ys(n) & p(n+1) \end{bmatrix}$$

with $p(n) = \sum_{k=1}^{n} \binom{n+k-2}{n-k} y^{k-1}$ *and* $s(n) = \sum_{k=1}^{n} \binom{n+k-1}{n-k} y^{k-1}$.

Lemma 5.2 allows us to simplify the expression (5.1) to have

$$\begin{aligned}
&\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ y & 1+y \end{bmatrix}^{n-2} \begin{bmatrix} y \\ y^2 \end{bmatrix} x^n = \\
&= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} p(n-2) & s(n-2) \\ ys(n-2) & p(n-1) \end{bmatrix} \begin{bmatrix} y \\ y^2 \end{bmatrix} x^n \\
&= \Big(yp(n-2) + ys(n-2) + y^2 s(n-2) + y^2 p(n-1) \Big) x^n \\
&= \left(yp(n-1) + y\left(s(n-2) + yp(n-1) \right) \right) x^n \\
&= p(n) x^n y \\
&= \sum_{k=1}^{n} \binom{n+k-2}{n-k} y^k x^n
\end{aligned}$$

By Theorem 5.1 we have

$$d(n,k) = \binom{n+k-2}{n-k}.$$

Now we turn our attention to the number of all possible divisions, i.e. we wish to determine the number $d(n)$. To do that, we again use a matrix $H(y)$ and Theorem 5.1. By setting $y = 1$ we have $H(1) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} F_1 & F_2 \\ F_2 & F_3 \end{bmatrix}$. Again, the following claim is easily guessed and verified by induction.

Lemma 5.3. $H(1)^n = \begin{bmatrix} F_{2n-1} & F_{2n} \\ F_{2n} & F_{2n+1} \end{bmatrix}.$

Finally, by Lemma 5.3 we can simplify expression (5.1) to have

$$\begin{aligned} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} F_{2n-5} & F_{2n-4} \\ F_{2n-4} & F_{2n-3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} F_{2n-5} + F_{2n-4} & F_{2n-4} + F_{2n-3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} F_{2n-3} & F_{2n-2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= F_{2n-1}. \end{aligned}$$

By Theorem 5.1 we have $d(n) = F_{2n-1}$.

5.2 Chocolate bars

Transfer matrices can also be used to obtain the sequence $r_k(n)$ denoting the number of ways to divide a rectangular strip $2 \times n$ into k parts. In this case, we do not add square by square, but column by column. So, let T denote a division of a strip where squares in the last column are in the same part, and S a division where squares in the last column are in different parts.
For $n = 1$, we have the same case as $n = 2$ in a honeycomb strip, so this case is represented by a vector

$$\overrightarrow{q_1} = x \begin{bmatrix} y \\ y^2 \end{bmatrix}.$$

Similar to the honeycomb case, if we start with a division of type T and we wish to obtain another division of type T, we can do that either by appending two new squares to the same part as the squares of the last column, or we can let two new squares form a new part. Hence, the corresponding entry in the transfer matrix is $1 + y$. By doing a similar analysis for other cases, we obtain the transfer matrix

$$Q(y) = \begin{bmatrix} 1+y & 2+y \\ y(2+y) & (1+y)^2 \end{bmatrix}.$$

Again, y is a formal variable keeping track of the number of pieces. So, for a strip $2 \times n$, the coefficient by $x^n y^k$ in the expression

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1+y & 2+y \\ y(2+y) & (1+y)^2 \end{bmatrix}^{n-1} \begin{bmatrix} y \\ y^2 \end{bmatrix} x^n$$

represents the number of ways to divide a rectangular strip $2\times n$ into exactly k parts.
We conclude this section by mentioning that in both cases, we could have obtained the asymptotic behavior of numbers $d(n)$ and $r(n)$ by computing the leading eigenvalue of the corresponding transfer matrix.

6 Concluding remarks

In this paper, we have employed two different methods to count divisions of narrow strips of squares and hexagons, respectively, into a given number of pieces, when cutting is allowed only along the edges of basic polygons. We have obtained several triangular integer arrays and determined formulas for their entries. Despite similar settings, the two problems behave in different ways: for honeycomb strips, the entries of the enumerating triangles are given as binomial coefficients with parameters dependent on the strip length and the number of pieces, while for chocolate bars, no closed-form expression has been obtained. We were able to show, though, that the entries in columns satisfy convolution-type recurrences with coefficients forming alternating rows of Pascal triangle.
Both problems were then addressed by using the transfer-matrix formalism. The original results for the total number of divisions were re-derived in a more compact way, demonstrating thus the power of the transfer-matrix method. However, we found the approach unsuitable for refining the aggregate results, for establishing the polynomial nature of columns, and for obtaining closed-form solutions in the rectangular case. Nevertheless, we believe that the transfer matrices would prove useful in treating several similar problems, as indicated by our experiments with wider strips in both square and hexagonal lattices and with narrow strips in the triangular lattice.

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