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Surprising bijections of the Riviera model

Tomislav Došlić, Mate Puljiz, Stjepan Šebek and Josip Žubrinić

Abstract

In this short note we give bijective proofs of two results from [3] which establish a correspondence between the maximal configurations in the Riviera model and two seemingly unrelated combinatorial objects: strongly restricted permutations and closed walks on a certain small graph. The results in the original paper were established by inspecting generating functions of the enumerating sequences.

1 Introduction

In [3] we introduced the *Riviera model* as a variant of the combinatorial settlement planning model first studied in [7, 8] and further developed in [2, 4]. The original problem, motivated by a real-life application and analyzed in [7, 8], proved to be too complex for analytical treatment, so we considered a 1-dimensional toy model which we were able to fully solve. Here, we briefly recall the setup. We begin with a finite strip of land divided into consecutive lots of equal size. Each lot can be occupied by a house or left vacant. The first requirement is that each occupied lot must be adjacent to at least one vacant lot. This resembles a Mediterranean settlement along the coast (hence the name 'Riviera model') stretching in the east-west direction, where each house is exposed to sunlight from the south and at least one

(Tomislav Došlić) University of Zagreb Faculty of Civil Engineering, Zagreb, Croatia and Faculty of Information Studies, Novo Mesto, Slovenia, tomislav.doslic@grad.unizg.hr

(Mate Puljiz) University of Zagreb Faculty of Electrical Engineering and Computing, Zagreb, Croatia, mate.puljiz@fer.unizg.hr

(Stjepan Šebek) University of Zagreb Faculty of Electrical Engineering and Computing, Zagreb, Croatia, stjepan.sebek@fer.unizg.hr

(Josip Žubrinić) University of Zagreb Faculty of Electrical Engineering and Computing, Zagreb, Croatia and Technische Universität Dortmund, Fakultät für Mathematik, Dortmund, Germany, josip.zubrinic@fer.unizg.hr

additional side (east or west depending on which side the adjacent vacant lot is). We can concisely write such configurations as binary words with ones denoting occupied and zeros vacant lots. The configurations satisfying this first requirement are called *permissible*. An example of a permissible configuration is:

$$110110001010011.$$

Note that the two houses on the left and right edge of the strip are assumed to be exposed to sunlight from west and east respectively. The second requirement is that the configurations be *maximal*, meaning that no further houses can be added to the vacant lots without violating the first insolation requirement. Clearly, the configuration above is not maximal, but the following is:

$$110110101011011.$$

It is clear that the first (permissibility) requirement is equivalent to asking that no three consecutive ones appear anywhere in the configuration. And it is not hard to see that the second (maximality) requirement can be verified by inspecting substrings of length 4 of the given configuration. To be precise, the following lemma was proven in [3].

Lemma 1.1 (Lemma 2.1 in [3]). *Let $n \in \mathbb{N}$. A configuration in the Riviera model is maximal if and only if, when padded with zeros, it does not contain any of the following (decorated*[1]*) substrings:*

$$1\underline{1}1, \quad 0\underline{0}0, \quad 01\underline{0}0, \quad 0\underline{0}10. \tag{1.1}$$

Next, with Lemma 1.1 in hand, the standard transfer matrix method approach was employed in [3] to establish a one-to-one correspondence between maximal Riviera configurations and walks on the directed graph in Figure 1. This digraph was created by taking all the allowed (i.e. not forbidden) substrings of length 3 and adding a directed edge from $u_1u_2u_3$ to $v_1v_2v_3$ if and only if they *overlap progressively*, meaning that $u_2 = v_1$ and $u_3 = v_2$; and if the substring $u_1u_2u_3v_3 = u_1v_1v_2v_3$ is not forbidden. The shaded starting/ending nodes were determined based on the boundary condition, which in our case states that the edge lots receive sunlight from the boundary side. By standard methods, see [3], one can now write the transfer matrix associated to this graph and obtain the bivariate generating function of the

[1]When inspecting whether a configuration $c_1 \dots c_n$ contains a decorated word $d_1 \dots \underline{d_k} \dots d_l$, we check against a padded word $\dots 000c_1 \dots c_n000 \dots$ but with the underlined letter of the decorated word aligned with c_i for $i = 1, \dots, n$. This is necessary as e.g. the configuration 10011 would otherwise be considered allowed (not containing any of the forbidden substrings), although it is not maximal.

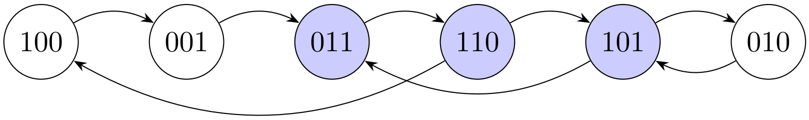


Figure 1: (Figure 3 in [3]) Transfer digraph $\mathcal{G}_{\mathcal{R}}$ for the Riviera model. For example, a maximal configuration 110011010110 is represented by the walk: $110 \to 100 \to 001 \to 011 \to 110 \to 101 \to 010 \to 101 \to 011 \to 110$. Each walk must start and end at shaded nodes.

sequence $(J_{k,n})$ enumerating maximal Riviera configurations of prescribed length n and occupancy k:

$$g(x,y) = \frac{1 + xy - (x - x^2)y^2 + x^2y^3 - x^3y^5}{1 - xy^2 - x^2y^3 - x^2y^4 + x^3y^6} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} J_{k,n}x^ky^n. \quad (1.2)$$

Remark 1.2. Setting $x = 1$ in the expression (1.2) above, one obtains $g(1,y) = f(y)$, the generating function for the sequence (f_n) which counts the number of all maximal configurations of length n:

$$f(y) = \frac{1 + y + y^3 - y^5}{1 - y^2 - y^3 - y^4 + y^6} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} J_{k,n}\right)y^n = \sum_{n=0}^{\infty} f_ny^n. \quad (1.3)$$

Similarly, by setting $y = 1$ in (1.2), one obtains $g(x,1) = h(x)$, the generating function for the sequence (h_k) which counts the number of maximal configurations (of variable length) with a fixed number k of occupied lots:

$$h(x) = \frac{1 + 2x^2 - x^3}{1 - x - 2x^2 + x^3} = \sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty} J_{k,n}\right)x^k = \sum_{k=0}^{\infty} h_kx^k. \quad (1.4)$$

By expanding the bivariate generating function $g = g(x,y)$, we obtain the precise distribution of the occupancies of maximal configurations relative to their length. The first few coefficients in the expansion of $g(x,y)$ are given in Table 1. Note that the ratio $\frac{k}{n}$ for non-zero coefficients stabilizes between $\frac{1}{2}$ and $\frac{2}{3}$ as n becomes large.
The first several values of the sequence (f_n), associated with the generating function $f(y)$, can be obtained as column sums of Table 1:

$$1, 1, 1, 3, 3, 4, 6, 9, 12, 16, 24, 33, 46, 64, \ldots$$

Similarly, the first several values of the sequence (h_k), associated with the generating function $h(x)$, can be obtained as row sums of Table 1:

$$1, 1, 5, 5, 14, 19, 42, 66, 131, \ldots$$

Table 1: The coefficients of $x^k y^n$ in the expansion of the bivariate generating function $g(x, y)$.

$k\backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	1																	
1		1																
2			1	3	1													
3					2	3												
4						1	6	6	1									
5								3	10	6								
6									1	10	20	10	1					
7											4	22	30	10				
8												1	15	49	50	15	1	
9														5	40	91	70	15
10															1	21	100	168

Upon consulting The On-Line Encyclopedia of Integer Sequences (OEIS) [9], we realized that these sequences had been already studied in quite different settings. The sequence (f_n) appears as OEIS sequence A080013 and is described as counting certain strongly restricted permutations. The sequence (h_n) appears as OEIS sequence A096976 and is described as counting closed walks on the path graph P_3 with a loop. The equipotency of the pairs of these combinatorial objects follows directly from the fact that their generating functions match (up to a finite degree polynomial). However, it is possible to construct explicit bijections demonstrating these equipotencies. This is the content of the next two sections.

2 The Riviera model and strongly restricted permutations

The notion of strongly restricted permutations was introduced by Lehmer in [5]. If W is some fixed subset of integers, one would like to count the number of all the permutations $\pi \in S_n$[2] such that $\pi(i) - i \in W$, for all $i \in [n]$.

In [10, Examples 4.7.9, 4.7.17–18] two techniques are presented for obtaining the generating function for the number of strongly restricted permutations for some particular sets W, namely the transfer-matrix method and the technique using factorization in free monoids.

In [1], Baltić devised a new technique for counting restricted permutations in case $\min W = -k$ and $\max W = r$ for some positive integers $k \leq r$. When $W = \{-2, -1, 2\}$, the sequence counting the corresponding restricted permutations of length n appears in OEIS under the number A080013. The

[2] S_n denotes the set of all permutations of the set $[n] = \{1, \dots, n\}$.

generating function of that sequence is $\dfrac{1-y^2}{1-y^2-y^3-y^4+y^6}$. Note that

$$\begin{aligned}\frac{1-y^2}{1-y^2-y^3-y^4+y^6} &= 1+y^3+y^4\cdot\frac{1+y+y^3-y^5}{1-y^2-y^3-y^4+y^6}\\ &= 1+y^3+y^4\cdot f(y),\end{aligned}$$

where $f(y)$ is the generating function for the number of maximal Riviera configurations of fixed length n, see (1.3). From here, the following result is immediate.

Theorem 2.1. *The number of maximal configurations of length n in the Riviera model is equal to the number of permutations π of length $n+4$, which satisfy the constraint*

$$\pi(i)-i\in\{-2,-1,2\}. \tag{2.1}$$

It turns out that one can construct a natural bijection between these two types of objects. The idea is to encode restricted permutations as walks on some digraph, similar to the one in Figure 1. If those two graphs are isomorphic, this isomorphism would automatically produce a bijection between the underlying combinatorial objects.
To construct this digraph we, once again, use the transfer-matrix method. One can argue as in [10, Example 4.7.9] to show that the method is applicable in this case. Let $\pi\in S_n$ be a permutation for which $\pi(i)-i\in W=\{-2,-1,2\}$, for all $i\in[n]$. One can rewrite such a permutation as a sequence of symbols in W. In order to check that such a sequence $u_1\dots u_n$ corresponds to a valid permutation, it suffices to check all the substrings of length 5. This is because the function $\sigma\colon[n]\to[n]$ defined as $\sigma(i)=i+u_i$ will be a permutation as soon as it is onto; and for this, one only needs to check whether $i\in\{\sigma(i-2),\sigma(i-1),\sigma(i),\sigma(i+1),\sigma(i+2)\}$, for all $3\le i\le n-2$. Additionally, one needs to check that 1 and 2 are in the set $\{\sigma(1),\sigma(2),\sigma(3),\sigma(4)\}$, and that $n-1$ and $n-2$ are in the set $\{\sigma(n-3),\sigma(n-2),\sigma(n-1),\sigma(n)\}$. The effect of this being that the walks must start and end at a certain subset of vertices of the constructed digraph. From here, one can write Algorithm 1 that produces this digraph which is an induced subgraph of the de Bruijn graph over the set of all 5 letter words in the alphabet $\{-2,-1,2\}$.
The digraph $\mathcal{G}$ constructed in Algorithm 1 has the vertex set $\mathcal{V}$ consisting of 30 allowed words of length 5. It turns out that this graph can be further condensed to give a smaller representation of our strongly restricted permutations. If one considers all the 4-letter words $\{-2,-1,2\}^4$ that do not appear as substrings of the 30 allowed words, one gets 59 forbidden words of

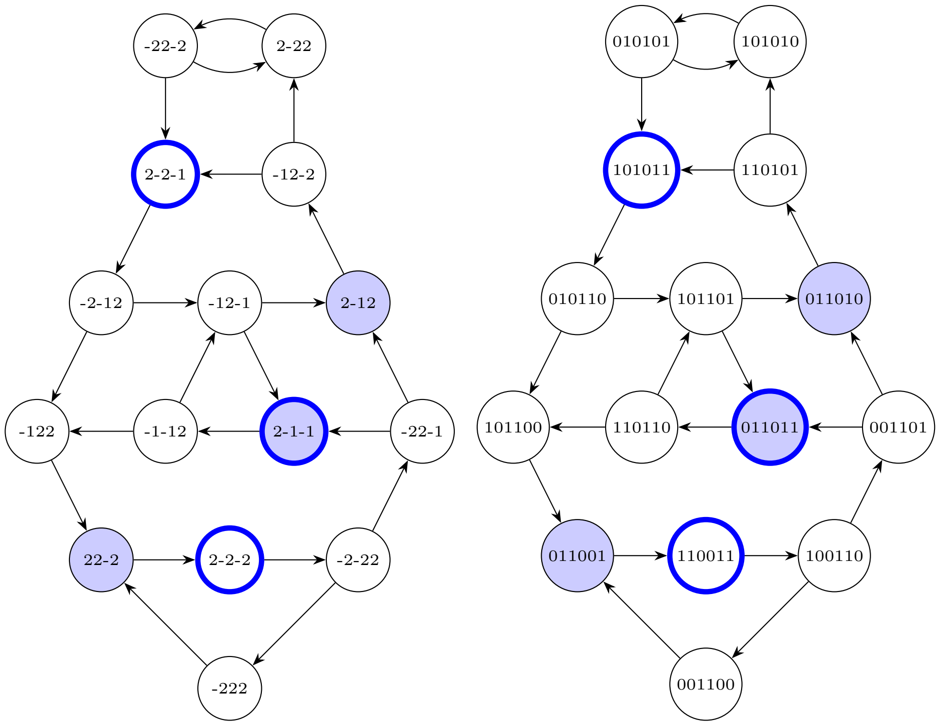
Algorithm 1 The creation of the digraph $\mathcal{G}$ for strongly restricted permutations

```
AllowedNodes = ∅
StartNodes = ∅
EndNodes = ∅
for u1u2u3u4u5 ∈ {−2, −1, 2}^5 do
    if 3 ∈ {1 + u1, 2 + u2, 3 + u3, 4 + u4, 5 + u5} then
        add node u1u2u3u4u5 to AllowedNodes
        if 1, 2 ∈ {1 + u1, 2 + u2, 3 + u3, 4 + u4, 5 + u5} then
            add node u1u2u3u4u5 to StartNodes
        end if
        if 4, 5 ∈ {1 + u1, 2 + u2, 3 + u3, 4 + u4, 5 + u5} then
            add node u1u2u3u4u5 to EndNodes
        end if
    end if
end for

E = ∅
for u1u2u3u4u5, v1v2v3v4v5 ∈ AllowedNodes do
    if u2u3u4u5 = v1v2v3v4 then
        add edge u1u2u3u4u5 → v1v2v3v4v5 to E
    end if
end for

V = ∅
for u1u2u3u4u5 ∈ AllowedNodes do
    if there is a path starting in StartNodes, passing through u1u2u3u4u5
and ending in EndNodes then
        add node u1u2u3u4u5 to V
    end if
end for

remove from E all the edges not involving nodes in V
return G = (V, E)
```



(a) (b)

Figure 2: The digraph $\mathcal{G}_{\mathcal{P}}$ in 2(a) encodes strongly restricted permutations satisfying the constraint (2.1). The starting nodes are shaded and thicker outlines indicate the ending nodes. The digraph $\mathcal{G}'_{\mathcal{R}}$ in 2(b) encodes configurations of the Riviera model using substrings of length 6. The nodes corresponding to the highlighted nodes in 2(a) via the unique digraph isomorphism are shaded and outlined in this graph too.

length 4. By inspection, one can check that each of the $213 = 3^5 - 30$ forbidden 5-letter words contains one of the 4-letter forbidden words which means that the same information contained in $\mathcal{G}$ can be encoded in a digraph with a vertex set consisting of only $22 = 3^4 - 59$ 4-letter words. Finally, if we use edges to encode allowed words, rather than just taking the whole induced subgraph of the corresponding de Bruijn graph, we can condense this digraph even further, and obtain the digraph in Figure 2(a) with 15 nodes representing allowed 3-letter words and an edge from $u_1u_2u_3$ to $v_1v_2v_3$ if and only if $u_1u_2u_3v_3 = u_1v_1v_2v_3$ is allowed 4-letter word. The highlighted nodes are either starting or ending nodes, or, in one case, both.

We would now like to match the digraph in Figure 2(a), call it $\mathcal{G}_{\mathcal{P}}$, with the digraph in Figure 1, call it $\mathcal{G}_{\mathcal{R}}$. Unfortunately, they are not isomorphic, but we can try to create higher edge graphs from the digraph $\mathcal{G}_\mathcal{R}$, details below, which encode the same information as $\mathcal{G}_\mathcal{R}$ — in hope of obtaining a graph isomorphic to $\mathcal{G}_\mathcal{P}$. This process is opposite of 'condensation' we have performed to the digraph produced by the Algorithm 1 in order to obtain the digraph $\mathcal{G}_\mathcal{P}$.

We have already noted that the graph $\mathcal{G}_\mathcal{R}$ shown in Figure 1 is a subgraph of the 3-dimensional de Bruijn graph over the alphabet $\{0,1\}$. We can construct a subgraph of the n-dimensional de Bruijn digraph, where $n > 3$, over the same alphabet, which will encode the same information as $\mathcal{G}_\mathcal{R}$. It turns out that $n = 6$ will do. The vertex set of this, so called, higher edge graph $\mathcal{G}'_\mathcal{R}$ consists of all the allowed words of length 6, which one can think of as all the possible walks of length 3 on the graph $\mathcal{G}_\mathcal{R}$. A directed edge from $c_1 \dots c_6$ to $d_1 \dots d_6$ is added to the edge set of $\mathcal{G}'_\mathcal{R}$ if and only if the corresponding words overlap progressively ($c_2 \dots c_6 = d_1 \dots d_5$). The digraph $\mathcal{G}'_\mathcal{R}$ is, therefore, the vertex-induced subgraph of the corresponding 6-dimensional de Bruijn graph. For more details on construction of higher edge graphs, see [6, Definition 2.3.4].

The graph $\mathcal{G}'_\mathcal{R}$ obtained by the above procedure, is shown in Figure 2(b). Note that it is isomorphic to $\mathcal{G}_\mathcal{P}$, and that this isomorphism is unique. Also note that the set of nodes at which the walks on $\mathcal{G}'_\mathcal{R}$ would be allowed to start and end is much larger than the set highlighted in Figure 2(b). More precisely, any node $c_1 \dots c_6$ for which $c_1c_2c_3 \in \{110, 101, 011\}$ would be a starting node, and if $c_4c_5c_6 \in \{110, 101, 011\}$, it would be an ending node. But the walks of length $n+1$ on $\mathcal{G}'_\mathcal{R}$ would then account for all the maximal configurations in the Riviera model of length $n+7$ — and that is not what we want, since the walks of length $n+1$ on $\mathcal{G}_\mathcal{P}$ encode the strictly restricted permutations of $[n+4]$.

If we consider the walks on $\mathcal{G}'_\mathcal{R}$ which start and end at the nodes that correspond to starting and ending nodes in $\mathcal{G}_\mathcal{P}$, we immediately note that all the configurations obtained in such a way always start with 0110 and end with 011. Using the graph $\mathcal{G}_\mathcal{R}$ in Figure 1 it is clear that adding the prefix 0110 and suffix 011 to a maximal configuration, again produces a 7-blocks longer (permissible) maximal configuration. This is because from each starting node, there is a backward path (going along edges in the direction opposite to the arrow direction) of length 4 which produces the prefix 0110; also from each ending node, there is a 3-step continuation of path which produces the suffix 011. Conversely, removing that same prefix and suffix from a maximal configuration of length $n+7$, produces a maximal configuration of length n. We can again argue using the graph $\mathcal{G}_\mathcal{R}$. Any walk starting with $011 \to 110$ after three steps must again reach one of the starting nodes; and walk ending in 011 when traced backwards must, after three steps going backwards, reach one of the ending nodes. This shows that there is a bijective correspondence between all the maximal Riviera configurations of length n and the maximal Riviera configurations of length $n+7$ starting with 0110 and ending with 011 which in turn are in a bijective correspondence with the strongly restricted permutations of length $n+4$. The bijection is obtained by translating walks on $\mathcal{G}_{\mathcal{P}}$ to walks on $\mathcal{G}'_{\mathcal{R}}$ and the other way around.

It is, in fact, possible to specify this bijection even more concisely, circumventing the graphs in Figure 2 altogether. Compare each edge in $\mathcal{G}_{\mathcal{R}}$ with all its associated edges in $\mathcal{G}'_{\mathcal{R}}$ and note that the corresponding edges in graph $\mathcal{G}_{\mathcal{P}}$ all represent adding the same symbol at the end. E.g. the transition $011 \to 110$ in $\mathcal{G}_{\mathcal{R}}$ corresponds to transitions $101011 \to 010110$, $011011 \to 110110$ and $110011 \to 100110$ in $\mathcal{G}'_{\mathcal{R}}$ and all of them in $\mathcal{G}_{\mathcal{P}}$ correspond to adding the letter 2 at the end. Collecting all this information together, we can label the edges of the graph $\mathcal{G}_{\mathcal{R}}$ in Figure 1 with the appropriate letter which is being added in the permutation graph $\mathcal{G}_{\mathcal{P}}$ corresponding to that transition. This edge-labeled graph is given in Figure 3.

We now summarize how to bijectively map any maximal Riviera configuration of length n to a strongly restricted permutation of length $n+4$ using Figure 3. Take any such maximal configuration and prefix it with 0110 and suffix it with 011. Then take a walk over the graph in Figure 3 (which will be of length $n+7-3=n+4$) and collect the labels $u_1 \dots u_{n+4}$ of all the edges traversed. Finally, construct the bijection $\sigma\colon [n+4] \to [n+4]$ as $\sigma(i) = i + u_i$ for $i \in [n+4]$.

As an example, the maximal configuration 10110 is first enlarged to the maximal configuration 0110|10110|011. Next, we examine the unique walk determined by this configuration: $011 \xrightarrow{2} 110 \xrightarrow{-1} 101 \xrightarrow{2} 010 \xrightarrow{-2} 101 \xrightarrow{-1} 011 \xrightarrow{2} 110 \xrightarrow{2} 100 \xrightarrow{-2} 001 \xrightarrow{-2} 011$. This walk generates the permutation σ encoded with the string 2-12-2-122-2-2, which is the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 5 & 2 & 4 & 8 & 9 & 6 & 7 \end{pmatrix}.$$

We end this section with a remark which will prove useful in the next section.

Remark 2.2. Above, we have argued that taking any maximal configuration $c_1 \dots c_n$ and prefixing it with 0110 and suffixing it with 011 yields a bijection between all the maximal Riviera configurations of length n and the maximal Riviera configurations of length $n+7$ starting with 0110 and ending with 011.

If we further add prefix 10 and suffix 001 to these already extended configurations, we obtain a bijective correspondence between the maximal Riviera

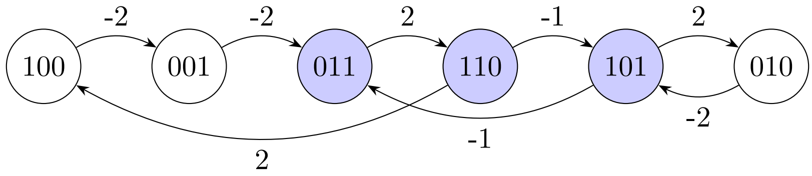


Figure 3: The digraph $\mathcal{G}_{\mathcal{R}}$ with labeled edges which encodes the bijection between maximal configurations of length n in the Riviera model and strongly restricted permutations of length $n+4$ where $W = \{-2, -1, 2\}$.

configurations $c_1 \dots c_n$ of length n and the configurations of length $n+12$ starting with 100110 and ending with 011001 which, although not maximal (because of the boundary condition), do not properly contain[3] any substrings forbidden by Lemma 1.1. Each of those extended configurations can, therefore, be represented as a walk on $\mathcal{G}_{\mathcal{R}}$ (Figure 1) starting at the node 100 and ending at the node 001. Conversely, if a configuration $100110c_1 \dots c_n011001$ (of length $n+12$) does not properly contain any substrings forbidden by Lemma 1.1, or equivalently, can be represented as a walk on $\mathcal{G}_{\mathcal{R}}$ (of length $n+9$) starting at 100 and ending at 001, then after removing prefix 100110 and suffix 011001 one is left with a proper maximal configuration $c_1 \dots c_n$ of length n. This is because removing prefix and suffix corresponds to cutting off the first part of the walk 100-001-011-110-10c_1-0c_1c_2 and the last part of the walk $c_{n-1}c_n$0-c_n01-011-110-100-001. Note that regardless of what c_1, c_2, c_{n-1}, and c_n are — the next node after $0c_1c_2$ as well as the node just before $c_{n-1}c_n0$ will always have to be one of the starting/ending nodes, which means that the remaining part of the walk encodes a proper maximal configuration $c_1c_2 \dots c_{n-1}c_n$.

3 The Riviera model and closed walks on P_3 with a loop

In (1.4) we have derived the generating function $h(x)$ for the number of maximal Riviera configurations (of variable length) containing a fixed number k of occupied lots. This sequence appears in OEIS [9] in two instances as A052547 with offset 3 and as A096976 with offset 5. There are three more related sequences: A006053, A028495, and A096975, satisfying the same recurrence relation with different initial conditions. Each of these sequences is connected to the number of walks on the graph P_3 (the path graph over

[3]By properly contain we mean contained as a substring within an unpadded configuration.

(a) P_3 with a loop, $\mathcal{P}$ for short (b) $\mathcal{P}$ with labeled nodes

Figure 4

three nodes) with a loop added at one of the end nodes. This graph is represented in Figure 4(a) and we denote it by $\mathcal{P}$. The precise connection relating this graph with our sequence is given in the following theorem.

Theorem 3.1. *The number of maximal Riviera configurations containing exactly k occupied lots is equal to the number of closed walks of length $k+4$ on the graph $\mathcal{P}$ which start and end at the node of degree 1. There is a natural bijection relating these quantities.*

Remark 3.2. If equipotency result is all one wishes to pursue, it suffices to compare the two sequences' generating functions. Note that

$$\begin{aligned}\frac{1-x-x^2}{1-x-2x^2+x^3} &= 1+x^2+x^4+x^4\cdot\frac{1+2x^2-x^3}{1-x-2x^2+x^3}\\ &= 1+x^2+x^4+x^4\cdot h(x),\end{aligned}$$

where $\frac{1-x-x^2}{1-x-2x^2+x^3}$ is the generating function of the sequence A096976 and $h(x)$ is the generating function for the number of maximal Riviera configurations (of variable length) containing a fixed number k of occupied lots, see (1.4).

Proof of Theorem 3.1. From Lemma 1.1 we know that no three consecutive 0's are allowed in a maximal configuration. That means that each two neighboring 1's must be separated by zero, one or two 0's. This further means that, after ignoring leading and trailing 0's, each maximal configuration can be identified by a sequence of strings in the set $\{11, 101, 1001\}$. We assume here that the last 1 in one string overlaps with the first 1 in the next. E.g. we would split the configuration 11011001101 as 11-101-11-1001-11-101.
From Lemma 1.1 we also see that 11 cannot be followed or preceded by 11 (as this would produce 111); 101 and 1001 cannot be followed or preceded by 1001 (as this would produce 0100 or 0010). It is easy to see that the remaining transitions: 1001-11 and 11-101 going in either direction, and the

loop at 101 — can all appear in a maximal configuration and are, thus, all allowed. These transitions are shown in the node-labeled graph $\mathcal{P}$ in Figure 4(b). We use undirected edges as in each case the transitions going either way are allowed.

Consider now the mapping which to each maximal Riviera configuration $c_1 \dots c_n$ assigns the configuration $100110c_1 \dots c_n011001$. By Remark 2.2 we know that this map is a bijection from the set of all maximal Riviera configurations with exactly k occupied lots to the set of configurations of the form $100110c_1 \dots c_n011001$ which have exactly $k+6$ occupied lots. Those obtained configurations are not maximal but do not properly contain substrings forbidden by Lemma 1.1.

Now each of those configurations of the form $100110c_1 \dots c_n011001$, where $c_1 \dots c_n$ is a proper maximal configuration with k occupied lots, can be represented as a walk of length $k+4$ on the graph $\mathcal{P}$ in Figure 4(b) which starts and ends at 1001. Conversely, one easily checks (by inspecting all length 2 walks) that a walk on this graph can never produce a configuration properly containing a substring which is forbidden by Lemma 1.1. Therefore, each walk of length $k+4$ starting and ending at 1001 will necessarily produce a configuration of the form $100110c_1 \dots c_n011001$ which does not properly contain a substring forbidden by Lemma 1.1 and has $k+6$ 1's. By Remark 2.2, the word $c_1 \dots c_n$ will be a proper maximal configuration with exactly k 1's.

Putting everything together gives us the required bijection. A maximal Riviera configuration $c_1 \dots c_n$ containing exactly k occupied lots is written as the string $100110c_1 \dots c_n011001$, which is then represented as a walk of length $k+4$ over the graph $\mathcal{P}$. As an example, the maximal configuration 10110 is mapped to $10110 \to 100110|10110|011001$ which corresponds to the walk: $1001 \to 11 \to 101 \to 101 \to 11 \to 1001 \to 11 \to 1001$ which begins and ends with the node 1001. □

4 Concluding remarks

Modern enumerative combinatorics uses a wide and evergrowing repertoire of methods and techniques. Prominent among them are analytical methods, based on extracting the information encoded in generating functions of the considered sequences. Another example are the methods based on transfer matrices. Powerful and useful as they are, in the eye of many practitioners they both suffer from a serious shortcoming: They do not reveal the true nature of connections between different combinatorially interesting families. According to their opinion, only a combinatorial, bijective, proof can offer deep(er) insight. So, combinatorial proofs are often sought even long after

interesting results have been obtained in other ways.
In this short note we provide combinatorial proofs for two enumerative results from our earlier paper on settlement planning [3]. We have constructed bijections connecting jammed configurations in a 1-dimensional toy model of settlement planning with, on one hand, restricted permutations, and, on the other hand, walks on a small graph. Hence, they reveal hidden connections between seemingly unrelated families of objects, providing support for the above claim and justifying the effort invested in their construction.
Our toy model, simple as it is, still offers many interesting problems which are still unanswered. For example, it would be interesting to (numerically) simulate its progression inwards, once the sea-front is fully developed (i.e., when no new constructions are legally possible). Another direction, more suitable for analytical and/or combinatorial treatment, would be to remain in one dimension but to allow multi-storied buildings and to investigate whether this modification results in more efficient jammed configurations. We are confident that the interested reader will find also problems and research questions we have overlooked.

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