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A Note on the r-Stirling Numbers of the First Kind

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Abstract

We study a generalization of Stirling numbers characterized by a property of having elements 1 through r in different cycles of a permutation. Motivated by the known explicit formulae for the Stirling numbers of the first kind, we derive an explicit expression for the r-Stirling numbers. In addition, we establish an alternating sum identity involving these sequences of numbers.

Keywords: Stirling numbers, Pascal type array, permutation, binomial coefficients

1 Introduction

The Stirling number of the first kind, denoted by ${n \brack k}$, is the number of permutations of the set $\{1, 2, \ldots, n\}$ with k disjoint cycles [7]. By placing a condition on certain elements, we obtain a generalized version of these numbers. For a positive integer r, the r-Stirling number of the first kind, denoted by ${n \brack k}_r$, is the number of permutations of the set $\{1, 2, \ldots, n\}$ with k disjoint cycles such that the elements $1, 2, \ldots, r$ appear in distinct cycles. Note that for $r = 1$, this definition recovers the Stirling numbers of the first kind, that is,

$$\begin{bmatrix} n \\ k \end{bmatrix}_1 = \begin{bmatrix} n \\ k \end{bmatrix}.$$

An explicit expression for $k = r$ follows directly from the definition, i. e. the number of permutations in the set $\{1, 2, \ldots, n\}$ with r disjoint cycles

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such that the elements $1, 2, \ldots, r$ are in distinct cycles is equal to

$$\begin{bmatrix} n \\ r \end{bmatrix}_r = \frac{(n-1)!}{(r-1)!}. \tag{1.1}$$

Namely, we begin by placing the elements $1, 2, \ldots, r$ into r distinct cycles

$$\underbrace{(1 \ \cdots \ )(2 \ \cdots \ ) \quad \cdots \quad (r \ \cdots \ )}_{r}.$$

Next, we insert the remaining elements one by one. For element $r+1$, there are r possible positions. After $r+1$ is placed, to place element $r+2$ there are $r+1$ possibilities. Continuing in the same fashion, each new element has one more position available than the previous one, up to the element n, which has $n-1$ possible positions. By the product rule, the total number of such permutations is

$$r \cdot (r+1) \cdots (n-1) = \frac{(n-1)!}{(r-1)!}.$$

Furthermore, r-Stirling numbers of the first kind satisfy a recurrence relation

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_r = \begin{bmatrix} n \\ k-1 \end{bmatrix}_r + n \begin{bmatrix} n \\ k \end{bmatrix}_r \tag{1.2}$$

with

$$\begin{bmatrix} n \\ k \end{bmatrix}_r = 0 \text{ for } n < r$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_r = \delta_{k,r} \text{ for } n = r,$$

where $\delta_{k,r} =$ if $k \neq r$ and $\delta_{k,r} = 1$ if $k = r$. This recurrence follows by the same counting argument as the recurrence for the Stirling numbers of the first kind.

Using the recurrence relation (1.2), one can easily compute the values of $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_r$ for various values of n, k, and r. Tables 1 and 2 present the first few numbers for $r = 2$ and $r = 3$, respectively.

Santmyer introduced curious sequences of rational numbers and used them to derive relations between Stirling numbers of the first kind and generalized harmonic numbers [11]. Benjamin, Preston, and Quinn studied the set $\mathcal{T}_n$ of permutations of the numbers 1 through n into two disjoint, nonempty cycles. By enumerating this set, they obtained identities for Stirling numbers of the

$n\backslash k$	2	3	4	5	6	7
2	1	0	0	0	0	0
3	2	1	0	0	0	0
4	6	5	1	0	0	0
5	24	26	9	1	0	0
6	120	154	71	14	1	0
7	720	1044	580	155	20	1

Table 1: The 2-Stirling numbers of the first kind.

$n\backslash k$	3	4	5	6	7	8
3	1	0	0	0	0	0
4	3	1	0	0	0	0
5	12	7	1	0	0	0
6	60	47	12	1	0	0
7	360	342	119	18	1	0
8	2520	2754	1175	245	25	1

Table 2: The 3-Stirling numbers of the first kind.

first kind having the lower index $k = 2$. In particular, they proved that for $n \geq 1$,

$$\sum_{k=1}^{n} k \begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix},$$

providing a bijection between $\mathcal{T}_n$ and sets of the permutations into various numbers of cycles [2].

Pan studied Pascal-type arrays that generalize Stirling numbers of the first kind and Jacobi-Stirling numbers. The author presented a convolution identity for these sequences [10]. It is well known that Stirling numbers with a small value of the lower index k can be expressed by means of harmonic and hyperharmonic numbers. In particular,

$$\begin{bmatrix} n \\ 2 \end{bmatrix} = (-1)^n (n-1)! H_{n-1}$$

where $H_n := 1 + 1/2 + \cdots + 1/n$. Kowalenko followed this topic and presented the first 10 expressions of the Stirling numbers of the first kind in terms of the generalized harmonic numbers [9]. Recall that hyperharmonic numbers appear in the book by Conway and Guy [6].

There are several known generalizations of Stirling numbers. Carlitz introduced the weighted Stirling numbers [5]. The r-Stirling numbers are a special case of generalized Stirling numbers studied by Hsu and Shine [8]. Broder gave the generating functions for the r-Stirling numbers [4],

$$\begin{aligned}
\sum_{k=r}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_r x^k &= x^r \prod_{j=r}^{n-1} (x+j), \\
\sum_{n=k}^{\infty} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r \frac{x^n}{n!} &= \frac{1}{k!}\Big(\frac{1}{1-x}\Big)^r \Big[\ln\Big(\frac{1}{1-x}\Big)\Big]^k, \\
\sum_{k=0}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r \frac{x^n}{n!} t^k &= \Big(\frac{1}{1-x}\Big)^{r+1}.
\end{aligned}$$

Benjamin, D. Geabler, and R. Geabler have shown that harmonic numbers can be expressed in terms of r-Stirling numbers, which leads to many identities [1].

In what follows, we present an explicit formula for r-Stirling numbers in Proposition 2.1. We prove it employing combinatorial arguments. Our main result is presented in Section 3. Using mathematical induction, we give alternating sum identities in full generality.

2 Explicit formulae for small parameters

It is well known that there is a double-sum explicit formula for the Stirling numbers of the first kind,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{j=n}^{2n-k} \binom{j-1}{k-1} \binom{2n-k}{j} \sum_{l=0}^{j-n} \frac{(-1)^{l+n-k} l^{j-k}}{l!(j-n-l)!}$$

(there is no known single-sum formula, as there is for Stirling numbers of the second kind). The best-known simpler explicit formulas are those having the value of the lower index either small or close to the value of the upper index.

The following identities can be derived directly by enumerating permutations.

$$\begin{aligned}
\begin{bmatrix} n \\ n-1 \end{bmatrix} &= \binom{n}{2}, \\
\begin{bmatrix} n \\ n-2 \end{bmatrix} &= \frac{3n-1}{4}\binom{n}{3}, \\
\begin{bmatrix} n \\ n-3 \end{bmatrix} &= \binom{n}{2}\binom{n}{4}, \\
\begin{bmatrix} n \\ n-4 \end{bmatrix} &= \frac{15n^3-30n^2+5n+2}{48}\binom{n}{5}.
\end{aligned}$$

One can also apply the Newton-Girard formula for symmetric polynomials to get expressions for Stirling numbers with small values of the lower index. For $n \in \mathbb{N}$ we have

$$\begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)!H_{n-1}. \tag{2.1}$$

Furthermore, it holds true

$$\begin{aligned}
\begin{bmatrix} n \\ 3 \end{bmatrix} &= \frac{(n-1)!}{2}\Big((H_{n-1})^2 - H_{n-1}^{(2)}\Big), \\
\begin{bmatrix} n \\ 4 \end{bmatrix} &= \frac{(n-1)!}{6}\Big((H_{n-1})^3 - 3H_{n-1}H_{n-1}^{(2)} + 2H_{n-1}^{(3)}\Big),
\end{aligned}$$

where $H_n^{(m)} := \sum_{k=1}^n 1/k^m$. The recurrence for these expressions,

$$(m+1)\begin{bmatrix} n \\ m+2 \end{bmatrix} = \sum_{k=0}^{m} \begin{bmatrix} n \\ m-k+1 \end{bmatrix} H_{n-1}^{(k+1)},$$

is given by Shen [12].

There is a nice proof for the relation (2.1) by a counting argument. We consider the set of permutations of length $n+1$ with two cycles and address the question of what number of permutations that have r as the smallest element in the right cycle is, where $2 \le r \le n+1$. Thus, we have

$$(1 \ \cdots \ \cdots \ )(r \ \cdots \ \cdots \ )$$

as the structure of the permutations. Having known that r is the smallest element in the right cycle, we conclude that the left cycle contains numbers $1, 2, \ldots, r-1$. Therefore, there are

$$(r-2)!$$

possibilities to arrange these numbers. Now, the element $r+1$ can be either in the left or the right cycle. There are r possibilities to place it, since $r+1$ can be placed to the right side of any of the elements 1 through r. Furthermore, we have $r+1$ possibilities to arrange the element $r+2$. In the same manner, one concludes that there are

$$r(r+1)(r+2)\cdots n = \frac{n!}{(r-1)!}$$

ways to arrange numbers $r+1, \ldots, n$.
Finally, by the product rule, there are

$$(r-2)!\frac{n!}{(r-1)!} = \frac{n!}{r-1}$$

considered permutations. Summation on r completes the proof,

$$\begin{aligned} \begin{bmatrix} n+1 \\ 2 \end{bmatrix} &= \sum_{r=2}^{n+1} \frac{n!}{r-1} \\ &= n! \sum_{r=2}^{n+1} \frac{1}{r-1} \\ &= n! \sum_{k=1}^{n} \frac{1}{k} \\ &= n! H_n. \end{aligned}$$

Proposition 2.1. *For positive integers n, we have the following explicit formula for the r-Stirling numbers:*

$$\begin{bmatrix} n \\ n-1 \end{bmatrix}_r = \frac{(n-r)(n-1+r)}{2}.$$

Proof. We consider the set of permutations of the set $\{1, \ldots, r, \ldots, n\}$ with $n-1$ cycles such that the elements $1, \ldots, r$ are in distinct cycles, that is

$$\underbrace{(1 \;\cdots\;) \quad \cdots \quad (r \;\cdots\;)(\;\cdots\;) \quad \cdots \quad (\;\cdots\;)}_{n-1}.$$

Out of $n-1$ cycles, r are reserved for the elements $1, 2, \ldots, r$, each in its own cycle. Thus, there are $n-1-r$ remaining cycles. To form these remaining cycles, we choose $n-1-r$ elements from the remaining $n-r$ elements, which can be done in $n-r$ ways, leaving one element unplaced. We denote this remaining element by a.

Now, a can be inserted into one of the existing cycles, either into one of he r cycles with element $1, \ldots, r-1$ or r, or into one of the remaining ones. In the first case, there are r possibilities.
In the other case, a can be placed into the remaining $n-1-r$ cycles. Let x denote the element currently in the cycle into which a is placed. Since inserting a into the cycle of x is equivalent to inserting x into the cycle of a, such configuration is counted twice. Therefore, this contributes $\frac{n-1-r}{2}$ valid placements.
Therefore, the total number of such permutations is equal to

$$(n-r)\left(r+\frac{n-1-r}{2}\right)=\frac{(n-r)(n-1+r)}{2}.$$

□

3 The alternating sum

The number of permutations of n elements with an even number of cycles is equal to the number of those with an odd number of cycles. Namely, in the standard notation, the element 2 is either in the first cycle or it is the first element in the second cycle. This provides the involution

$$\begin{gathered}(1\, a_1 \quad \cdots \quad a_j\, 2\, b_1 \quad \cdots \quad b_k)(\ \cdots\ ) \quad \cdots \quad (\ \cdots\ )\\ \uparrow\downarrow \\ (1\, a_1 \quad \cdots \quad a_j)(2\, b_1 \quad \cdots \quad b_k)(\ \cdots\ ) \quad \cdots \quad (\ \cdots\ )\end{gathered}$$

which changes the parity of a permutation [3]. This reasoning proves that the alternating sum of Stirling numbers with the upper index n is equal to zero,

$$\sum_{k=1}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} = 0.$$

We now present an alternating sum identity involving r-Stirling numbers of the first kind and provide a proof using strong induction.

Theorem 3.1. *For positive integers n and $r \geq 2$, r-Stirling numbers of the first kind satisfy*

$$\sum_{k=r}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_r = (-1)^r \frac{(n-2)!}{(r-2)!}. \tag{3.1}$$

Proof. The relation (3.1) holds true for $n = r$, since the right hand side of the equation gives $(-1)^r$ which is equal to

$$(-1)^r \begin{bmatrix} r \\ r \end{bmatrix}_r .$$

Moreover, for $n = r + 1$, the left-hand side of the equation gives

$$(-1)^r \begin{bmatrix} r+1 \\ r \end{bmatrix}_r + (-1)^{r+1} \begin{bmatrix} r+1 \\ r+1 \end{bmatrix}_r = (-1)^r (r-1)$$

by relation (1.1), while the right-hand side of the equation gives

$$(-1)^r \frac{(r+1-2)!}{(r-2)!} = (-1)^r (r-1).$$

Furthermore, by applying the recurrence relation (1.2), we have

$$\begin{aligned}
&\sum_{k=r}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_r \\
&= (-1)^r \begin{bmatrix} n \\ r \end{bmatrix}_r + (-1)^{r+1} \begin{bmatrix} n \\ r+1 \end{bmatrix}_r + \cdots + (-1)^{n-1} \begin{bmatrix} n \\ n-1 \end{bmatrix}_r + (-1)^n \begin{bmatrix} n \\ n \end{bmatrix}_r \\
&= (-1)^r (n-1) \begin{bmatrix} n-1 \\ r \end{bmatrix}_r + (-1)^{r+1} \left(\begin{bmatrix} n-1 \\ r \end{bmatrix}_r + (n-1) \begin{bmatrix} n-1 \\ r+1 \end{bmatrix}_r \right) + \cdots \\
&+ (-1)^{n-1} \left(\begin{bmatrix} n-1 \\ n-2 \end{bmatrix}_r + (n-1) \begin{bmatrix} n-1 \\ n-1 \end{bmatrix}_r \right) + (-1)^n \begin{bmatrix} n-1 \\ n-1 \end{bmatrix}_r \\
&= (n-2) \sum_{k=r}^{n-1} (-1)^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_r \\
&= (n-2)(-1)^r \frac{(n-1-2)!}{(r-2)!} \\
&= (-1)^r \frac{(n-2)!}{(r-2)!}.
\end{aligned}$$

□

For $r = 2$, identity (3.1) simplifies to an elegant form

$$\sum_{k=2}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_2 = (n-2)!.$$

Identity (3.1) simplifies to an (almost equally) elegant form for $r = 3$.

Example 3.2. For $r = 4$ and $n = 7$, according to Theorem 3.1 we have following alternating summation

$$\begin{aligned}\sum_{k=4}^{7}(-1)^k \begin{bmatrix}7\\k\end{bmatrix}_4 &= 120 - 74 + 15 - 1\\ &= 60 \left(= (-1)^4\frac{(7-2)!}{(4-2)!}\right).\end{aligned}$$

4 Concluding remarks

The r-Stirling numbers of the first kind are a natural generalization of the Stirling numbers of the first kind, having a similar combinatorial interpretation. The r-Stirling numbers of the first kind are characterised by the requirement that elements 1 through r are in different cycles of a permutation.

Stirling numbers of the first kind, which have the value of the lower index either small or close to the value of the upper index, have known explicit formulas. In this paper, we aimed to find an explicit expression for the r-Stirling numbers as well. It remains to derive other explicit expressions for these numbers. In particular, it would be interesting to answer the question on the recursive nature of such formulas.

Being motivated by the involution which gives the alternating sum of Stirling numbers of the first kind, we consider the analogous problem for the r-Stirling numbers. We presented a family of identities with alternating sign for the rows of triangular matrices of r-Stirling numbers of the first kind. It would be valuable to find a combinatorial proof of these identities. Furthermore, one could find other summations involving these numbers or extend the problem to other generalizations as well as to r-Stirling numbers of the second kind.

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