Proceedings of the 5th Croatian Combinatorial Days
September 19–20, 2024

ISBN: 978-953-8168-77-2
DOI: 10.5592/CO/CCD.2024.05

A New Theorem from the Number Theory and its Application for a 3-adic Valuation for Large Schröder Numbers

Jovan Mikić

Abstract

By using central Delannoy numbers and a new theorem from number theory, we give almost a complete 3-adic valuation of large Schröder numbers S_n, except for the case S_{6k+4}, $k \geq 0$.

Keywords: Large Schröder number, Central Delannoy number, 3-adic valuation, Catalan number, Kummer's theorem.
2020 ***Mathematics Subject Classification***: 05A10, 11B65.

1 Introduction

The large Schröder Numbers S_n count all lattice paths in the plane from $(0,0)$ to (n,n) by using horizontal steps $(1,0)$, vertical steps $(0,1)$, and diagonal steps $(1,1)$ such that never rise above the main diagonal $y = x$. It is known [6, Exercise 6.39], at least, 11 combinatorial objects are counted by large Schröder numbers S_n.
A formula for calculating large Schröder numbers S_n is also well-known:

$$S_n = \sum_{k=0}^{n} \frac{1}{k+1} \binom{n+k}{k} \binom{n}{k}. \tag{1.1}$$

This sequence starts with: $S_0 = 1$, $S_1 = 2$, $S_2 = 6$, $S_3 = 22$, $S_4 = 90$, $S_5 = 394$, $S_6 =$1806, . . .; and it can be found as sequence $A006318$ in [5]
Recently, a new formula for a 3-adic valuation of large Schröder numbers S_n has been discovered, where the p-adic valuation of an integer $y \geq 0$ is

(Jovan Mikić) Faculty of Technology, Faculty of Natural Sciences and Mathematics, University of Banja Luka, Bosnia and Herzegovina, jovan.mikic@tf.unibl.org

definied as follows: Let p be a prime number, and let y be an integer. Then $v_p(y)$ denotes the exponent of the highest power of the prime number p that divides y, i.e., $p^{v_p(y)}$ divides y, $p^{v_p(y)+1}$ does not divide y, and is called the p-adic valuation of y.
It is readily verified that $v_3(S_0) = v_3(S_1) = v_3(S_3) = v_3(S_5) = 0$, $v_3(S_2) = v_3(S_6) = 1$, and $v_3(S_4) = 2$.
Furthermore, it is well-known that Catalan number C_n counts all lattice paths in the plane from $(0,0)$ to (n,n) by using horizontal steps $(1,0)$ and vertical steps $(0,1)$ that never rise above the main diagonal $y = x$. Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ represent the famous sequence which has the most applications in combinatorics after the binomial coefficients.
This sequence starts with: $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$, $C_5 = 42$, $C_6 = 132$, $\ldots$; and it can be found as sequence $A000108$ in [5]. It is readily verified that $v_3(C_0) = v_3(C_1) = v_3(C_2) = v_3(C_3) = v_3(C_4) = 0$, and $v_3(C_5) = v_3(C_6) = 1$.
Now, we are ready to present a recently [4, Theorem 6, Eqns. (28) and (29), p. 6] discovered formula for a 3-adic valuation of large Schröder numbers S_n:

Theorem 1.1. *Let n be a non-negative integer. Then the following formulae*

$$v_3(S_{2n+1}) = v_3(C_n), \tag{1.2}$$

$$v_3(S_{2n+2}) = 1 + v_3(2n+1) + v_3(C_n). \tag{1.3}$$

hold.

The Equation (1.2) represents, by our opinion, one of the most beautiful results in number theory. It tells us that large Schröder numbers with odd indices have the same factorization of powers of three as Catalan numbers C_n.
Recently, the proof of Theorem 1.1 was given using the little Schröder numbers s_n. The little Schröder numbers s_n count, among other things, the number of plane trees with a given set of leaves, the number of ways of inserting parentheses into a sequence, and the number of ways of dissecting a convex polygon into smaller polygons by inserting diagonals. Furthermore, if n is a natural integer, then it is known that $s_n = \frac{1}{2} \cdot S_n$.
This sequence starts with: $s_0 = 1$, $s_1 = 1$, $s_2 = 3$, $s_3 = 11$, $s_4 = 45$, $s_5 = 197$, $s_6 = 903$, $\ldots$; and it can be found as sequence $A001003$ in [5]. Obviously, $v_3(s_n) = v_3(S_n)$, for any non-negative integer n.
The aim of this paper is to give an another proof of Theorem 1.1. We give almost a complete proof of Theorem 1.1 by using central Delannoy numbers D_n and a new theorem from number theory.

The central Delannoy numbers D_n represent the number of lattice paths in the plane from $(0,0)$ to (n,n) by using horizontal steps $(1,0)$, vertical steps $(0,1)$, and diagonal steps $(1,1)$. Such paths are also known in the literature as royal paths. Sulanke [7] gave, in a recreational spirit, a collection of 29 configurations counted by these numbers.

For calculating central Delannoy numbers D_n, there are, at least, two formulae [1, Eq. (1), p. 1]:

$$D_n = \sum_{k=0}^{n} \binom{n}{k}^2 2^k, \tag{1.4}$$

$$D_n = \sum_{k=0}^{n} \binom{n+k}{k} \binom{n}{k}. \tag{1.5}$$

This sequence starts with: $D_0 = 1$, $D_1 = 3$, $D_2 = 13$, $D_3 = 63$, $D_4 = 321$, $D_5 = 1683$, $D_6 = 8989$, $\ldots$; and it can be found as sequence $A001850$ in [5]. It is readily verified that $v_3(D_0) = v_3(D_2) = v_3(D_6) = 0$, $v_3(D_1) = v_3(D_4) = 1$, and $v_3(D_3) = v_3(D_5) = 2$.

It is known [4, Eq. 5, p. 2] that central Delannoy numbers satisfy the following second order recurrence relation:

$$(n+2)D_{n+2} = 3(2n+3) \cdot D_{n+1} - (n+1) \cdot D_n. \tag{1.6}$$

A connection between central Delannoy numbers and large Schröder numbers is given [4, Eq. 6, p. 2] by the following formula:

$$S_n = \frac{1}{2} \cdot (-D_{n-1} - D_{n+1} + 6D_n), \tag{1.7}$$

where n is a natural number.

Finally, we present a recently discovered new theorem from number theory. Let x and y be integers such that $x \neq 0$ and $x \neq \pm 1$. Let $\omega_x(y)$ denote the exponent of the highest power of an integer x that divides an integer y. Note that we shall use the notation $v_p(y)$ instead of $\omega_x(y)$ when the integer x is equal to a prime number p.

Let n be a non-negative integer. Let a and b be integers such that are relatively prime and $a \neq -b$, as well as $a + b \neq \pm 1$.

The recently discovered theorem [4, Theorem 1, Eqns. (18) and (19)] from number theory states that

Theorem 1.2.

$$\omega_{a+b}(\sum_{k=0}^{2n}\binom{2n}{k}^2\cdot a^{2n-k}\cdot b^k)=\omega_{a+b}(\binom{2n}{n}), \tag{1.8}$$

$$\omega_{a+b}(\sum_{k=0}^{2n+1}\binom{2n+1}{k}^2\cdot a^{2n+1-k}\cdot b^k)=1+\omega_{a+b}((2n+1)\cdot\binom{2n}{n}). \tag{1.9}$$

By setting $a := 2$ and $b := 1$ in (1.8) and (1.9), we obtain the formulae for calculating a 3-adic valuation of central Delannoy numbers [4, Theorem 5, Eqns. (27) and (26), p.6], where $v_3(y) = \omega_3(y)$, as we said before (see also [2]). Therefore, the following two formulae are true for any non-negative integer n:

$$v_3(D_{2n})=v_3(\binom{2n}{n}), \tag{1.10}$$

$$v_3(D_{2n+1})=1+v_3(2n+1)+v_3(\binom{2n}{n}). \tag{1.11}$$

Recently, Lengyel gave, among other, the formula [1, Theorem 10, p. 8] for 3-adic valuation of central Delannoy numbers. However, his formula is true for any n sufficiently large. Our two formulae from Eqns. (9) and (10) are true for any natural number n.
Lengyel also gave the formula [1, Theorem 17, p. 19] for the 3-adic valuation of large Schröder numbers S_n for the case $n-1$ is divisible by 3.
The rest of the paper is structured, as follows: in the second section, we present auxiliary results. In the third section, we give a proof of (1.2). In the fourth section, we give a proof of (1.3) for the cases $n-2$ is divisible by 3 and n is divisible by 3.

2 Auxiliary Results

Our first auxiliary result is:

Proposition 2.1. *Let n be a natural number. Then the following recurrence relation holds:*

$$2(n+1)S_n = 3D_n - D_{n-1}. \tag{2.1}$$

The second auxiliary result is:

Proposition 2.2. *Let n be a non-negative integer. Then the following recurrence relation holds:*

$$2(2n+1)S_n = D_{n+1} - D_{n-1}. \tag{2.2}$$

Finally, the third auxiliary result is:

Proposition 2.3. *Let k be a non-negative integer. Then the two following equations hold:*

$$v_3(\binom{6k+2}{3k+1}) = v_3(\binom{6k}{3k}) = v_3(\binom{2k}{k}) \tag{2.3}$$

$$v_3(\binom{6k+4}{3k+2}) = 1 + v_3(2k+1) + v_3(\binom{2k}{k}) \tag{2.4}$$

Proofs of (2.1) and (2.2) easily follow from (1.6) and (1.7). Furthermore, the proof of (2.3) follow from Kummer's theorem [3]. Namely, we recall that Kummer's theorem states that for given integers $n \geq m \geq 0$ and a prime number p, the p-adic valuation of the binomial coefficient $\binom{n}{m}$ is equal to the number of carryovers that occur when m and $n-m$ are added in base p.
Finally, the proof of (2.4) follows from (2.3) and by setting $n = 3k+1$ in the well-known equation:

$$\binom{2n+2}{n+1} = 2(2n+1)C_n; \tag{2.5}$$

where n is a non-negative integer.
Obviously, by the definition of Catalan numbers, it follows that

$$v_3(C_n) = v_3(\binom{2n}{n}) - v_3(n+1).$$

Therefore, we leave the proofs of the auxiliary results to the readers.

3 A Proof of the Eq. (1.2)

Substituting $2n+1$ for n in (2.1), we get

$$4(n+1)S_{2n+1} = 3D_{2n+1} - D_{2n}, \tag{3.1}$$

where n is a non-negative integer.
From (3.1) it follows that

$$v_3(n+1) + v_3(S_{2n+1}) = v_3(3D_{2n+1} - D_{2n}). \tag{3.2}$$

By (1.10) and (1.11), we know that

$$v_3(3 \cdot D_{2n+1}) = 1 + v_3(D_{2n+1}) > 1 + v_3(D_{2n}) > v_3(D_{2n}). \tag{3.3}$$

From (3.3) it follows that

$$v_3(3D_{2n+1} - D_{2n}) = v_3(D_{2n}). \tag{3.4}$$

Furthermore, by (3.4), (3.2) becomes

$$v_3(n+1) + v_3(S_{2n+1}) = v_3(D_{2n}). \tag{3.5}$$

Finally, by (1.10), we have gradually:

$$v_3(n+1) + v_3(S_{2n+1}) = v_3(\binom{2n}{n}), \tag{3.6}$$

$$v_3(S_{2n+1}) = v_3(\binom{2n}{n}) - v_3(n+1), \tag{3.7}$$

$$v_3(S_{2n+1}) = v_3(\frac{1}{n+1} \cdot \binom{2n}{n}), \tag{3.8}$$

$$v_3(S_{2n+1}) = v_3(C_n). \tag{3.9}$$

The last equation proves the assertion (1.2).

4 A Proof of (1.3)

We give a proof of (1.3) for the cases $n-2$ is divisible by 3 and n is divisible by 3.
Substituting $2n+2$ for n in (2.2), we get

$$2(4n+5)S_{2n+2} = D_{2n+3} - D_{2n+1}, \tag{4.1}$$

where n is a non-negative integer.
By (4.1), it follows that

$$v_3(4n+5) + v_3(S_{2n+2}) = v_3(D_{2n+3} - D_{2n+1}). \tag{4.2}$$

The first case:
Let $n-2$ be divisible by 3. Then $n = 3k+2$, where k is a non-negative integer. Since $v_3((4(3k+2)+5) = v_3(12k+13) = 0$, we get from equation (4.2):

$$v_3(S_{6k+6}) = v_3(D_{6k+7} - D_{6k+5}). \tag{4.3}$$

By (1.11) and Kummer's theorem, it can be shown that

$$v_3(D_{6k+7}) = 1 + v_3(2k+1) + v_3(\binom{2k}{k}) - v_3(k+1). \tag{4.4}$$

Furthermore, by (1.11), it follows that

$$v_3(D_{6k+5}) = 2 + v_3(2k+1) + v_3(\binom{2k}{k}). \tag{4.5}$$

By Eqns. (4.4) and (4.5), it follows that $v_3(D_{6k+7}) < v_3(D_{6k+5})$. Therefore, we conclude that

$$v_3(D_{6k+7} - D_{6k+5}) = v_3(D_{6k+7}). \tag{4.6}$$

By using (4.4) and (4.6), (4.3) becomes

$$v_3(S_{6k+6}) = 1 + v_3(2k+1) + v_3(\binom{2k}{k}) - v_3(k+1). \tag{4.7}$$

By setting $n = 3k + 2$ in (1.3), the equation (1.3) gradually becomes

$$\begin{aligned} v_3(S_{6k+6}) &= 1 + v_3(6k+5) + v_3(C_{3k+2}), \\ &= 1 + v_3(\binom{6k+4}{3k+2}) - v_3(3k+3), \\ &= 1 + (1 + v_3(2k+1) + v_3(\binom{2k}{k})) - 1 - v_3(k+1), \\ &= 1 + v_3(2k+1) + v_3(\binom{2k}{k}) - v_3(k+1). \end{aligned} \tag{4.8}$$

By using (4.7) and (4.8), it follows that (1.3) is true if $n = 3k + 2$. This completes the proof of the first case.

The second case:

Let n be a non-negative integer divisible by 3. Then $n = 3k$, where $k \geq 0$. By setting $n = 3k$ in (4.1), we obtain that

$$v_3(S_{6k+2}) = v_3(D_{6k+3} - D_{6k+1}). \tag{4.9}$$

By using (1.11) and (2.4), it can be shown that

$$\begin{aligned} v_3(D_{6k+3}) &= 1 + v_3(6k+3) + v_3(\binom{6k+2}{3k+1}), \\ &= 2 + v_3(2k+1) + v_3(\binom{2k}{k}). \end{aligned} \tag{4.10}$$

On the other hand, by using (1.11) and (2.3), it can be shown that

$$v_3(D_{6k+1}) = 1 + v_3(\binom{2k}{k}). \tag{4.11}$$

By using (4.10) and (4.11), it follows that $v_3(D_{6k+1}) < v_3(D_{6k+3})$. Hence, we conclude that

$$v_3(D_{6k+3} - D_{6k+1}) = v_3(D_{6k+1}). \tag{4.12}$$

Therefore, by using (4.12), it follows that

$$v_3(S_{6k+2}) = 1 + v_3(\binom{2k}{k}). \tag{4.13}$$

By setting $n = 3k$ in (1.3), the equation (1.3) gradually becomes

$$\begin{aligned} v_3(S_{6k+2}) &= 1 + v_3(6k+1) + v_3(C_{3k}), \\ &= 1 + v_3(\binom{6k}{3k}) - v_3(3k+1), \\ &= 1 + v_3(\binom{2k}{k}). \end{aligned} \tag{4.14}$$

By using (4.13) and (4.14), it follows that (1.3) is true for $n = 3k$. This completes the proof of the second case.

Remark 4.1. The case $n-1$ is divisible by 3 of (1.3) must be treated with another approach (see, for example, [1, Theorem 17, p. 19] or [4, Section 12, p. 25]) due to the fact that

$$v_3(D_{6k+5}) = v_3(D_{6k+3}) = 1 + v_3(D_{6k+4}). \tag{4.15}$$

We conjecture that the following equation is true:

$$v_3(3D_{6k+4} - D_{6k+3}) = v_3(D_{6k+3}), \tag{4.16}$$

where $k \geq 0$.

Acknowledgments

I want to thank professor Tomislav Došlić for inviting me to the 5th Croatian Combinatorial Days in Zagreb. Also, I am grateful to the anonymous referee for valuable comments that improved the quality of this article.

References

[1] T. Lengyel. On some p-adic properties and supercongruences of Delannoy and Schröder numbers. *Integers*, 21:#A86, 2021.

[2] T. X. S. Li and A. F. Y. Zhao. Weighted free lattice paths and Legendre polynomials. *Integers*, 22:#A103, 2022.

[3] D. Mihet. Legendre's and Kummer's Theorems Again. *General article*, 2010.

[4] J. Mikić. On new divisibility properties of generalized central trinomial coefficients and Legendre polynomials. *Available at arxiv.org/pdf/2311.14623*, page 29 pages, 2023.

[5] N. J. A. Sloane. On-Line Encyclopedia of Integer Sequences (oeis). *Published electronically at http://oeis.org/*, 2024.

[6] R. P. Stanley. *Enumerative Combinatorics, Vol. 2.* Cambridge University Press, 1999.

[7] R. A. Sulanke. Objects counted by the central Delannoy numbers. *J. Integer Seq.*, 6:Article 03.1.5., 2003.