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On two sequences and their hypersequences

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Abstract

We study the hypersequences $\left(a_n^{(r)}\right)_{n\in\mathbb{N}_0}$ and $\left(b_n^{(r)}\right)_{n\in\mathbb{N}_0}$, $r \in \mathbb{N}_0$, of the two sequences $a_n := (-1)^n$ and $b_n := (-1)^{n+1}n, n \in \mathbb{N}_0$. First, we show the relationship between these hypersequences. Subsequently, we prove that both the rth rows and the nth columns of the arrays $\left(a_n^{(r)}\right)$ and $\left(b_n^{(r)}\right)$, $r, n \in \mathbb{N}_0$, satisfy linear recurrence relations. This yields alternative representations of $a_n^{(r)}$ and $b_n^{(r)}$. Finally, we determine their ordinary generating functions and the recurrence relations of two special subsequences.

Keywords: Hypersequences; recurrences; binomial coefficients; Stirling numbers of the first kind; ordinary generating functions; Catalan numbers.
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1 Introduction

The sequence $(c_n) = (0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots)$ is the sequence A001057 in the On-Line Encyclopedia of Integer Sequences (OEIS ®) [3]. It is the sequence of all integers and can be described as follows: start from 0 and go forward and backward with increasing step sizes. Accordingly, the sequence can be defined by

$$c_{2n} = -n, \quad c_{2n+1} = n + 1, \quad n \geq 0, \tag{1.1}$$

showing that the function $c\colon \mathbb{N}_0 \to \mathbb{Z}$, $n \mapsto (-1)^{n+1} \cdot \left\lfloor \frac{n+1}{2} \right\rfloor$ is a bijection. Since the successor function $s\colon \mathbb{N}_0 \to \mathbb{N}$, $n \mapsto n + 1$, is also a bijection, this shows that $\mathbb{N}$, $\mathbb{N}_0$, and $\mathbb{Z}$ have the same cardinality, namely $\aleph_0$, the first transfinite cardinal number.

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Lemma 1.1. *The sequence* $(c_n)_{n\in\mathbb{N}_0}$ *satisfies the recurrences*

$$c_0 = 0, \quad c_{n+1} = c_n + (n+1)(-1)^n, \quad n \geq 0, \tag{1.2}$$

and

$$c_0 = 0, \quad c_{n+1} = -c_n + \frac{1}{2}(1 + (-1)^n), \quad n \geq 0, \tag{1.3}$$

which have the solution

$$c_n = \sum_{k=0}^{n} (-1)^{k+1} k = \frac{1}{4}(1-(2n+1)(-1)^n) = 2\cdot\left(\left\lfloor \frac{n+1}{2} \right\rfloor\right)^2 - \binom{n+1}{2}, \tag{1.4}$$

for $n \geq 0$.

Proof. By definition, we have $c_0 = 0$. Let n be even, that is $n = 2m$, $m \in \mathbb{N}_0$. Then, by (1.1) we have $c_{n+1} = c_{2m+1} = m + 1$, and $c_n = c_{2m} = -m$. Therefore, $c_n + (n+1)(-1)^n = c_{2m} + (2m+1)(-1)^{2m} = -m + (2m+1) = m+1 = c_{2m+1} = c_{n+1}$. Now, let n be odd, that is $n = 2m+1$, $m \in \mathbb{N}_0$. Then, again by (1.1) we have $c_{n+1} = c_{2m+2} = -(m+1)$, and $c_n = c_{2m+1} = m+1$. Hence, $c_{n+1} - c_n = c_{2m+2} - c_{2m+1} = -(m+1) - (m+1) = -(2m+2) = (2m+2)\cdot(-1)^{2m+1} = (n+1)\cdot(-1)^n$, and these two cases prove (1.2).
Let n be even, that is $n = 2m$, $m \in \mathbb{N}_0$. Then, by (1.1) we have $c_{n+1} + c_n = c_{2m+1} + c_{2m} = m+1+(-m) = 1 = (1+(-1)^{2m})/2$. Similarly, for n odd, that is $n = 2m+1$, $m \in \mathbb{N}_0$, we have again by (1.1) $c_{n+1} + c_n = c_{2m+2} + c_{2m+1} = -(m+1) + m + 1 = 0 = (1 + (-1)^{2m+1})/2$, and this proves (1.3).
The solution of the recurrence (1.2) can be obtained by the method of backward substitution and noting that $c_0 = 0$

$$\begin{aligned}
c_n &= c_{n-1} - n\cdot(-1)^n \\
&= c_{n-2} - (n-1)\cdot(-1)^{n-1} - n\cdot(-1)^n \\
&\ \vdots \\
&= c_0 - 1\cdot(-1)^1 - 2\cdot(-1)^2 - \cdots - (n-1)\cdot(-1)^{n-1} - n\cdot(-1)^n \\
&= -\sum_{k=1}^{n}(-1)^k k = \sum_{k=0}^{n}(-1)^{k+1}k,
\end{aligned}$$

and this is the first formula of (1.4).
Adding Equations (1.2) and (1.3) (for $n-1$ instead of n), we get the second formula of (1.4).
Finally, let us evaluate $S_n := \sum_{k=0}^{n}(-1)^{k+1}k$. It is well-known that $T_n := \sum_{k=0}^{n} k = \binom{n+1}{2}$. Then $S_n + T_n = \sum_{k=0}^{n}\left(1+(-1)^{k+1}\right)k = 2\sum_{k=1}^{\lfloor\frac{n+1}{2}\rfloor}(2k-1) = 2\left(\lfloor\frac{n+1}{2}\rfloor\right)^2$. Solving this equation for S_n, we obtain the third formula of (1.4). $\square$

Remark 1.2. By (1.2) we obtain two alternative recurrences for (c_n), namely

$$c_0 = 0,\, c_1 = 1, \quad c_{n+2} = c_n - (-1)^n, \quad n \geq 0, \tag{1.5}$$

and

$$c_0 = 0,\, c_1 = 1,\, c_2 = -1, \quad c_{n+3} = -c_{n+2} + c_{n+1} + c_n, \quad n \geq 0, \tag{1.6}$$

because $c_{n+2} = c_{n+1} - (n+2)(-1)^n = c_n + (n+1)(-1)^n - (n+2)(-1)^n = c_n - (-1)^n$. This proves (1.5). Consequently, $c_{n+3} = c_{n+1} + (-1)^n$, $c_{n+2} = c_n - (-1)^n$. The recurrence relation (1.6) now follows by adding these two equations.

Note that the first formula on the right-hand side of (1.4) states that (c_n) is the sequence of partial sums of $b_n := (-1)^{n+1}n$, $n \in \mathbb{N}_0$, that is $(b_n) = (0, 1, -2, 3, -4, 5, -6, 7, -8, \ldots)$ (the sequence A181983) and that the sequence of nonnegative integers $(n) = (0, 1, 2, 3, 4, 5, \ldots)$ (the sequence A001477) is given by $((-1)^{n+1}b_n)_{n\in\mathbb{N}_0}$. Hence, in this paper we shall study the hypersequences of $(b_n)_{n\in\mathbb{N}_0}$ and those of the closely related sequence $a_n := (-1)^n$, $n \in \mathbb{N}_0$ (the sequence A033999).

2 Hypersequences of $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$

Let $(f_n)_{n\in\mathbb{N}_0}$ be an arbitrary sequence (of real or complex numbers). Then the *hypersequence of the rth generation* is defined recursively for all $r \in \mathbb{N}$ and $n \in \mathbb{N}_0$ as

$$f_n^{(r)} := \sum_{k=0}^{n} f_k^{(r-1)}, \quad \text{and} \quad f_n^{(0)} := f_n. \tag{2.1}$$

For $r = 1$, we have $f_n^{(1)} = \sum_{k=0}^{n} f_k^{(0)} = \sum_{k=0}^{n} f_k$ and this is the sequence of partial sums of $(f_n)_{n\in\mathbb{N}_0}$; for $r = 2$, we have $f_n^{(2)} = \sum_{k=0}^{n} f_k^{(1)} = \sum_{k=0}^{n}\left(\sum_{j=0}^{k} f_j\right)$ and this is the sequence of partial sums of $(f_n^{(1)})_{n\in\mathbb{N}_0}$, and so on.

By means of this definition we obtain the array $(f_n^{(r)})$, where $r \in \mathbb{N}_0$ is the row and $n \in \mathbb{N}_0$ is the column of this array (see Table 1).

The next theorem is well-known (see, e.g., [1, Proposition 2, p. 945] for the special case $f_0^i = f_0$ for all $i \in \{1, \ldots, r\}$). The second equation follows from the fact that $k \in \{0, 1, \ldots, n\}$ if and only if $n - k \in \{0, 1, \ldots, n\}$.

Theorem 2.1. *Let $(f_n)_{n\in\mathbb{N}_0}$ be an arbitrary sequence (of real or complex numbers) and $(f_n^{(r)})_{n\in\mathbb{N}_0}$, $r \in \mathbb{N}_0$, be the hypersequence of the rth generation as defined before. Then for all $r \in \mathbb{N}$ and $n \in \mathbb{N}_0$*

$$f_n^{(r)} = \sum_{k=0}^{n} \binom{n+r-1-k}{r-1} f_k = \sum_{k=0}^{n} \binom{r+k-1}{k} f_{n-k}. \tag{2.2}$$

$r\backslash n$	0	1	2
0	f_0	f_1	f_2
1	f_0	f_0+f_1	$f_0+f_1+f_2$
2	f_0	$2f_0+f_1$	$3f_0+2f_1+f_2$
3	f_0	$3f_0+f_1$	$6f_0+3f_1+f_2$
4	f_0	$4f_0+f_1$	$10f_0+4f_1+f_2$
5	f_0	$5f_0+f_1$	$15f_0+5f_1+f_2$

$r\backslash n$	3	4
0	f_3	f_4
1	$f_0+f_1+f_2+f_3$	$f_0+f_1+f_2+f_3+f_4$
2	$4f_0+3f_1+2f_2+f_3$	$5f_0+4f_1+3f_2+2f_3+f_4$
3	$10f_0+6f_1+3f_2+f_3$	$15f_0+10f_1+6f_2+3f_3+f_4$
4	$20f_0+10f_1+4f_2+f_3$	$35f_0+20f_1+10f_2+4f_3+f_4$
5	$35f_0+15f_1+5f_2+f_3$	$70f_0+35f_1+15f_2+5f_3+f_4$

Table 1: The hypersequences $(f_n^{(r)})_{n\in\mathbb{N}_0}, r\in\mathbb{N}_0$, of $(f_n)_{n\in\mathbb{N}_0}$

Applying this theorem to the sequences $f_n := a_n$ and $f_n := b_n$, $n\in\mathbb{N}_0$, we obtain the following results (see Table 2 and Table 3).

Corollary 2.2. *For all $r, n\in\mathbb{N}_0$:*

$$a_n^{(r)} = \sum_{k=0}^{n} \binom{n+r-1-k}{r-1}(-1)^k = \sum_{k=0}^{n} \binom{r+k-1}{k}(-1)^{n-k} \tag{2.3}$$

$$b_n^{(r)} = \sum_{k=0}^{n} \binom{n+r-1-k}{r-1}(-1)^{k+1}k = \sum_{k=0}^{n} \binom{r+k-1}{k}(-1)^{n-k+1}(n-k). \tag{2.4}$$

The next corollary gives the relationship between these hypersequences.

Corollary 2.3. *For all $r, n\in\mathbb{N}_0$:*

$$a_n^{(r)} = b_n^{(r)} + b_{n+1}^{(r)} \tag{2.5}$$

and, conversely,

$$b_n^{(r)} = \sum_{k=1}^{n}(-1)^{k+1}a_{n-k}^{(r)} = \sum_{k=0}^{n-1}(-1)^{n-k+1}a_k^{(r)}. \tag{2.6}$$

Proof. By (2.4) we have

$$\begin{aligned}
b_{n+1}^{(r)} &= \sum_{k=0}^{n+1} \binom{r+k-1}{k} (-1)^{n+2-k}(n+1-k) \\
&= \sum_{k=0}^{n+1} \binom{r+k-1}{k} (-1)^{n-k}(n-k) + \sum_{k=0}^{n+1} \binom{r+k-1}{k} (-1)^{n-k} \\
&= -\sum_{k=0}^{n} \binom{r+k-1}{k} (-1)^{n-k+1}(n-k) + \binom{r+n}{n+1} + \\
&\quad + \sum_{k=0}^{n} \binom{r+k-1}{k} (-1)^{n-k} - \binom{r+n}{n+1} \\
&= -b_n^{(r)} + a_n^{(r)},
\end{aligned}$$

and this proves (2.5).
Conversely, applying the method of backward substitution we obtain from (2.5)

$$\begin{aligned}
b_n^{(r)} &= -b_{n-1}^{(r)} + a_{n-1}^{(r)} \\
&= -\big(-b_{n-2}^{(r)} + a_{n-2}^{(r)}\big) + a_{n-1}^{(r)} = b_{n-2}^{(r)} - a_{n-2}^{(r)} + a_{n-1}^{(r)} \\
&= -b_{n-3}^{(r)} + a_{n-3}^{(r)} - a_{n-2}^{(r)} + a_{n-1}^{(r)} \\
&= \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&= (-1)^n b_0^{(r)} + \sum_{k=1}^{n} (-1)^{k+1} a_{n-k}^{(r)}.
\end{aligned}$$

Together with $b_0^{(r)} = 0$ this proves the second formula of (2.6). Finally, the first formula of (2.6) follows from the fact that $k \in \{1, 2, \ldots, n\}$ if and only if $n-k \in \{0, 1, \ldots, n-1\}$. ☐

This corollary shows that knowing $\big(b_n^{(r)}\big)_{n\in\mathbb{N}_0}$, we obtain $\big(a_n^{(r)}\big)_{n\in\mathbb{N}_0}$ from (2.5) and, conversely, knowing $\big(a_n^{(r)}\big)_{n\in\mathbb{N}_0}$, we obtain $\big(b_n^{(r)}\big)_{n\in\mathbb{N}_0}$ from (2.6). The hypersequences of the rth generation $\big(a_n^{(r)}\big)$ and $\big(b_n^{(r)}\big)$ satisfy the following recurrences:

Theorem 2.4. *For all $r \in \mathbb{N}_0$:*

$$a_0^{(r)} = 1, \quad a_{n+1}^{(r)} = -a_n^{(r)} + \binom{r+n}{n+1}, \quad n \geq 0, \tag{2.7}$$

$$b_0^{(r)} = 0, \quad b_1^{(r)} = 1, \quad b_{n+1}^{(r)} = -2b_n^{(r)} - b_{n-1}^{(r)} + \binom{r+n-1}{n}, \quad n \geq 1. \tag{2.8}$$

$r\backslash n$	0	1	2	3	4	5	6	7	8	9	10
0	1	-1	1	-1	1	-1	1	-1	1	-1	1
1	1	0	1	0	1	0	1	0	1	0	1
2	1	1	2	2	3	3	4	4	5	5	6
3	1	2	4	6	9	12	16	20	25	30	36
4	1	3	7	13	22	34	50	70	95	125	161
5	1	4	11	24	46	80	130	200	295	420	581
6	1	5	16	40	86	166	296	496	791	1211	1792
7	1	6	22	62	148	314	610	1106	1897	3108	4900

Table 2: The hypersequences of $(a_n)_{n\in\mathbb{N}_0}$

$r\backslash n$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	-2	3	-4	5	-6	7	-8	9	-10
1	0	1	-1	2	-2	3	-3	4	-4	5	-5
2	0	1	0	2	0	3	0	4	0	5	0
3	0	1	1	3	3	6	6	10	10	15	15
4	0	1	2	5	8	14	20	30	40	55	70
5	0	1	3	8	16	30	50	80	120	175	245
6	0	1	4	12	28	58	108	188	308	483	728
7	0	1	5	17	45	103	211	399	707	1190	1918

Table 3: The hypersequences of $(b_n)_{n\in\mathbb{N}_0}$

Proof. By (2.4) it follows that for all $n \geq 0$

$$
\begin{aligned}
a_{n+1}^{(r)} &= \sum_{k=0}^{n+1} \binom{r+k-1}{k} (-1)^{n+1-k} \\
&= -\sum_{k=0}^{n} \binom{r+k-1}{k} (-1)^{n-k} + \binom{r+n}{n+1} \\
&= -a_n^{(r)} + \binom{r+n}{n+1}.
\end{aligned}
$$

Together with $a_0^{(r)} = 1$ this proves the assertion (2.7).
By (2.5) and (2.7) it follows that $b_n^{(r)} = -b_{n-1}^{(r)} + a_{n-1}^{(r)}$ and $a_{n-1}^{(r)} = -a_n^{(r)} + \binom{r+n-1}{n}$ for all $n \geq 1$. Substituting the last equation into the first one, we get $b_n^{(r)} = -b_{n-1}^{(r)} - a_n^{(r)} + \binom{r+n-1}{n}$. By (2.5) it follows that $b_n^{(r)} = -b_{n-1}^{(r)} - (b_n^{(r)} + b_{n+1}^{(r)}) + \binom{r+n-1}{n}$. Solving this equation for $b_{n+1}^{(r)}$, we obtain (2.8) valid for all $n \geq 1$. The initial values for all $r \geq 0$ are by definition $b_0^{(r)} = 0$ and $b_1^{(r)} = \sum_{k=0}^{1} b_k^{(r-1)} = b_0^{(r-1)} + b_1^{(r-1)} = b_1^{(r-1)} = b_1^{(r-2)} = \cdots = b_1^{(0)} = 1$, and this proves (2.8). $\square$

We now derive alternative representations of $a_n^{(r)}$ and $b_n^{(r)}$.

Theorem 2.5. *For all $n \in \mathbb{N}_0$:*

$$
a_n^{(0)} = (-1)^n, \quad 2a_n^{(r+1)} = a_n^{(r)} + \binom{r+n}{n}, \quad r \geq 0, \tag{2.9}
$$

with the solution

$$
a_n^{(r)} = \frac{1}{2^r}\left((-1)^n + \sum_{k=0}^{r-1} 2^k \binom{n+k}{k} \right), \tag{2.10}
$$

and for $r \geq 1$

$$
\begin{aligned}
b_n^{(0)} &= (-1)^{n+1} n, \\
b_n^{(1)} &= \frac{1}{4}(1 - (2n+1)(-1)^n), \\
4b_n^{(r+1)} &= 4b_n^{(r)} - b_n^{(r-1)} + \binom{r+n-1}{n-1},
\end{aligned}
\tag{2.11}
$$

with the solution

$$
b_n^{(r)} = \sum_{k=0}^{n-1} (-1)^{n-k+1} a_k^{(r)} = \frac{(-1)^{n+1}}{2^r} \cdot \left(n + \sum_{j=0}^{r-1} 2^j \Big(\sum_{k=0}^{n-1} (-1)^k \binom{k+j}{j} \Big) \right). \tag{2.12}
$$

Proof. First, we derive a recurrence relation of $a_n^{(r)}$ with respect to r. By definition, we have $a_n^{(0)} = a_n = (-1)^n$. By (2.3) and by the addition formula for binomial coefficients ([2, Equation (5.8)]) it follows that for all $r \geq 0$

$$\begin{aligned} a_n^{(r+1)} &= \sum_{k=0}^{n} \binom{r+1+k-1}{k} (-1)^{n-k} \\ &= \sum_{k=0}^{n} \binom{r+k-1}{k} (-1)^{n-k} + (-1)^n \sum_{k=0}^{n} \binom{r+k-1}{k-1} (-1)^{n-k}. \end{aligned}$$

The first term on the right-hand side is by definition equal to $a_n^{(r)}$, whereas the second term is equal to

$$\begin{aligned} \sum_{k=1}^{n} \binom{r+k-1}{k-1} (-1)^{n-k} &= \sum_{k=0}^{n-1} \binom{r+k}{k} (-1)^{n-(k+1)} \\ &= -\sum_{k=0}^{n-1} \binom{r+1+k-1}{k} (-1)^{n-k}. \end{aligned}$$

Hence,

$$\begin{aligned} a_n^{(r+1)} &= a_n^{(r)} - \sum_{k=0}^{n-1} \binom{r+1+k-1}{k} (-1)^{n-k} \\ &= a_n^{(r)} - \sum_{k=0}^{n} \binom{r+1+k-1}{k} (-1)^{n-k} + \binom{r+n}{n} \\ &= a_n^{(r)} - a_n^{(r+1)} + \binom{r+n}{n}. \end{aligned}$$

By solving this equation for $a_n^{(r+1)}$, the statement (2.9) follows.
The solution of this recurrence relation can be obtained by the method of backward substitution. However, we simply check that the right-hand side of equation (2.10) satisfies (2.9). For $r = 0$ we get $(-1)^n$. Furthermore,

$$\begin{aligned} 2a_n^{(r+1)} &= 2 \cdot \frac{1}{2^{r+1}} \left((-1)^n + \sum_{k=0}^{r} 2^k \binom{n+k}{k} \right) \\ &= \frac{1}{2^r} \left((-1)^n + \sum_{k=0}^{r-1} 2^k \binom{n+k}{k} + 2^r \binom{n+r}{r} \right) \\ &= \frac{1}{2^r} \left((-1)^n + \sum_{k=0}^{r-1} 2^k \binom{n+k}{k} \right) + \binom{n+r}{r} \\ &= a_n^{(r)} + \binom{r+n}{n} \end{aligned}$$

by the symmetry property of the binomial coefficients. This proves the formula (2.10).
By (2.4) and by the addition formula for binomial coefficients it follows that for all $r \geq 0$

$$\begin{aligned} b_n^{(r+1)} &= \sum_{k=0}^{n} \binom{r+1+k-1}{k} (-1)^{n-k+1}(n-k) \\ &= \sum_{k=0}^{n} \binom{r+k-1}{k} (-1)^{n-k+1}(n-k) + \\ &\quad + \sum_{k=0}^{n} \binom{r+k-1}{k-1} (-1)^{n-k+1}(n-k), \end{aligned}$$

where the first term on the right-hand side is by definition equal to $b_n^{(r)}$, and the second term for k instead of $k-1$ is equal to

$$\begin{aligned} &\sum_{k=1}^{n} \binom{r+k-1}{k-1} (-1)^{n-k+1}(n-k) = \\ &= \sum_{k=0}^{n-1} \binom{r+k}{k} (-1)^{n-(k+1)+1}(n-(k+1)) \\ &= -\sum_{k=0}^{n-1} \binom{r+k}{k} (-1)^{n-k+1}(n-k) - \sum_{k=0}^{n-1} \binom{r+k}{k} (-1)^{n-k}. \end{aligned}$$

The first sum on the right-hand side is by definition equal to

$$-\sum_{k=0}^{n-1} \binom{r+k}{k} (-1)^{n-k+1}(n-k) = -\sum_{k=0}^{n} \binom{r+k}{k} (-1)^{n-k+1}(n-k) = b_n^{(r+1)},$$

whereas the second sum is equal to

$$\begin{aligned} -\sum_{k=0}^{n-1} \binom{r+k}{k} (-1)^{n-k} &= -\sum_{k=0}^{n} \binom{r+k}{k} (-1)^{n-k} + \binom{r+n}{n} \\ &= -a_n^{(r+1)} + \binom{r+n}{n}. \end{aligned}$$

Hence, we obtain

$$b_n^{(r+1)} = b_n^{(r)} - b_n^{(r+1)} - a_n^{(r+1)} + \binom{r+n}{n}$$

and solving for $b_n^{(r+1)}$

$$2b_n^{(r+1)} = b_n^{(r)} - a_n^{(r+1)} + \binom{r+n}{n}, \tag{2.13}$$

or, for $r-1$ instead of r

$$2b_n^{(r)} = b_n^{(r-1)} - a_n^{(r)} + \binom{r-1+n}{n}. \tag{2.14}$$

Subtracting Equation (2.14) from the double of Equation (2.13) we obtain

$$4b_n^{(r+1)} - 2b_n^{(r)} = 2b_n^{(r)} - b_n^{(r-1)} - 2a_n^{(r+1)} + a_n^{(r)} + 2\binom{r+n}{n} - \binom{r-1+n}{n}.$$

Finally, solving for $b_n^{(r+1)}$ and using the recurrence relation (2.9) and the addition formula for binomial coefficients, we obtain

$$\begin{aligned} 4b_n^{(r+1)} &= 4b_n^{(r)} - b_n^{(r-1)} - a_n^{(r)} - \binom{r+n}{n} + a_n^{(r)} + 2\binom{r+n}{n} - \binom{r-1+n}{n} \\ &= 4b_n^{(r)} - b_n^{(r-1)} + \binom{r+n-1}{n-1}. \end{aligned}$$

This recurrence relation is linear and of second order and has the initial values $b_n^{(0)} = (-1)^{n+1}n$ and by Lemma (1.1) $b_n^{(1)} = c_n = \frac{1}{4}(1-(2n+1)(-1)^n)$. This proves the assertion (2.11).

We now show that $g(r,n) := \sum_{k=0}^{n-1}(-1)^{n-k+1}a_k^{(r)}$ solves the recurrence (2.11). First, we have $g(0,n) = \sum_{k=0}^{n-1}(-1)^{n-k+1}a_k^{(0)} = \sum_{k=0}^{n-1}(-1)^{n-k+1}(-1)^k = (-1)^{n+1}n$ and since by (2.10) $a_n^{(1)} = \sum_{k=0}^{n}(-1)^k = \frac{1}{2}(1+(-1)^n)$ it follows that

$$\begin{aligned} g(1,n) &= \sum_{k=0}^{n-1}(-1)^{n-k+1}a_k^{(1)} = \frac{1}{2}(-1)^{n+1}\sum_{k=0}^{n-1}((-1)^k+1) \\ &= \frac{1}{2}(-1)^{n+1}\left(\frac{1}{2}(1+(-1)^{n-1})+n\right) = \frac{1}{4}(1-(2n+1)(-1)^n). \end{aligned}$$

Consequently, $g(r,n)$ satisfies the two initial conditions. Furthermore, by

(2.9) and by the addition formula of the binomial coefficients, we have

$$\begin{aligned}
&4g(r+1,n) - 4g(r,n) + g(r-1,n) = \\
&= \sum_{k=0}^{n-1}(-1)^{n-k+1}(4a_k^{(r+1)} - 4a_k^{(r)} + a_k^{(r-1)}) \\
&= \sum_{k=0}^{n-1}(-1)^{n-k+1}\Big(2\cdot(2a_k^{(r+1)} - a_k^{(r)}) - (2a_k^{(r)} - a_k^{(r-1)})\Big) \\
&= \sum_{k=0}^{n-1}(-1)^{n-k+1}\left(2\binom{r+k}{k} - \binom{r-1+k}{k}\right) \\
&= \sum_{k=0}^{n-1}(-1)^{n-k+1}\binom{r+k}{k} + \sum_{k=0}^{n-1}(-1)^{n-k+1}\binom{r+k-1}{k-1}
\end{aligned}$$

and the right-hand side is equal to $\binom{r+n-1}{n-1}$, since with k instead of $k-1$ the second sum can be expressed as follows:

$$\begin{aligned}
\sum_{k=0}^{n-1}(-1)^{n-k+1}\binom{r+k-1}{k-1} &= \sum_{k=0}^{n-2}(-1)^{n-k}\binom{r+k}{k} \\
&= -\sum_{k=0}^{n-2}(-1)^{n-k+1}\binom{r+k}{k}.
\end{aligned}$$

It follows that $g(r,n) = b_n^{(r)}$. Furthermore, by (2.10) it follows that

$$\begin{aligned}
b_n^{(r)} &= \sum_{k=0}^{n-1}(-1)^{n-k+1}a_k^{(r)} = \sum_{k=0}^{n-1}(-1)^{n-k+1}\cdot\frac{1}{2^r}\left((-1)^k + \sum_{j=0}^{r-1}2^j\binom{k+j}{j}\right) \\
&= \frac{1}{2^r}\sum_{k=0}^{n-1}(-1)^{n+1} + \frac{1}{2^r}\sum_{k=0}^{n-1}(-1)^{n-k+1}\sum_{j=0}^{r-1}2^j\binom{k+j}{j} \\
&= \frac{(-1)^{n+1}n}{2^r} + \frac{(-1)^{n+1}}{2^r}\sum_{k=0}^{n-1}(-1)^k\sum_{j=0}^{r-1}2^j\binom{k+j}{j} \\
&= \frac{(-1)^{n+1}}{2^r}\left(n + \sum_{j=0}^{r-1}2^j\Big(\sum_{k=0}^{n-1}(-1)^k\binom{k+j}{j}\Big)\right),
\end{aligned}$$

and this proves (2.12). □

Note that by (2.3) and (2.10) we have shown that for all r, $n \in \mathbb{N}_0$, we have

$$\sum_{k=0}^{n}\binom{r+k-1}{k}(-1)^{n-k} = \frac{1}{2^r}\left((-1)^n + \sum_{k=0}^{r-1}2^k\binom{n+k}{k}\right), \tag{2.15}$$

and by (2.4) and (2.12) we have

$$\begin{aligned}&\sum_{k=0}^{n}\binom{r+k-1}{k}(-1)^{n-k+1}(n-k)=\\&=\frac{(-1)^{n+1}}{2^r}\Big(n+\sum_{j=0}^{r-1}2^j\big(\sum_{k=0}^{n-1}(-1)^k\binom{k+j}{j}\big)\Big).\end{aligned}\tag{2.16}$$

By (2.10) the first few values of $a_n^{(r)}$ for fixed $r\geq 0$ are:

i) $r=0$: $a_n^{(0)}=(-1)^n$, (A033999)

ii) $r=1$: $a_n^{(1)}=\frac{1}{2}(1+(-1)^n)$, (A059841, characteristic function of even numbers)

iii) $r=2$: $a_n^{(2)}=\frac{1}{4}(1+(-1)^n)+\frac{1}{2}(n+1)$, (A004526$(n+1)$, nonnegative integers repeated)

iv) $r=3$: $a_n^{(3)}=\frac{1}{8}(1+(-1)^n)+\frac{1}{4}(n+1)(n+3)$, (A002620$(n+2)$)

v) $r=4$: $a_n^{(4)}=\frac{1}{16}(1+(-1)^n)+\frac{1}{24}(n+1)(n+3)(2n+7)$, (A002623)

vi) $r=5$: $a_n^{(5)}=\frac{1}{32}(1+(-1)^n)+\frac{1}{48}(n+1)(n+3)^2(n+5)$, (A001752).

In particular, setting $r=3$ and $n=2m$, $m\in\mathbb{N}_0$, in (2.15) gives the sequence of the square numbers A000290$(m+1)$

$$\sum_{k=0}^{2m}\binom{k+2}{2}(-1)^k=(m+1)^2,$$

while for $n=2m+1$, $m\in\mathbb{N}_0$, we get the sequence of the oblong numbers A002378$(m+1)$

$$\sum_{k=0}^{2m+1}\binom{k+2}{2}(-1)^{k-1}=(m+1)(m+2).$$

On the other hand, again by (2.10) the first few sequences of $a_n^{(r)}$ for fixed $n\geq 0$ are:

i) $n=0$: $a_0^{(r)}=1$, (A000012, the all 1's sequence)

ii) $n=1$: $a_1^{(r)}=r-1$, (A023443)

iii) $n=2$: $a_2^{(r)}=\frac{1}{2}(r^2-r+2)$, (A152947)

iv) $n=3$: $a_3^{(r)} = \frac{1}{6}(r^3+5r-6)$, (A283551)

v) $n=4$: $a_4^{(r)} = \frac{1}{24}(r^4+2r^3+11r^2-14r+24)$, (after removing the first term this is the sequence A223718)

vi) $n=5$: $a_5^{(r)} = \frac{1}{120}(r^5+5r^4+25r^3-5r^2+94r-120)$, (after removing the first two terms, this is the sequence A257890).

Remark 2.6. A look at the polynomials $n! \cdot a_n^{(r)}$ (for fixed $n \geq 0$) shows that their coefficients in descending powers of r are given by the triangle as shown in Table 4. This table is up to the sign of the columns for odd k given by the triangle A054651. By a slight modification of the formula given in A054651, these coefficients $U(n,k)$, $n,k \in \mathbb{N}_0$, are given by

$$U(n,k) = (-1)^k \sum_{i=0}^{k} \begin{bmatrix} i+n-k \\ n-k \end{bmatrix} \frac{n!}{(i+n-k)!}, \tag{2.17}$$

where $\left[{i+n-k \atop n-k}\right]$ is an unsigned Stirling number of the first kind. Hence, by (2.3) we have

$$\begin{aligned} \sum_{k=0}^{n} \binom{r+k-1}{k}(-1)^{n-k} &= \frac{1}{n!}\sum_{k=0}^{n} U(n,k) r^{n-k} \\ &= \sum_{k=0}^{n} \left((-1)^k \sum_{i=0}^{k} \begin{bmatrix} i+n-k \\ n-k \end{bmatrix} \frac{1}{(i+n-k)!} \right) r^{n-k}. \end{aligned} \tag{2.18}$$

Note that $U(n,n) = (-1)^n \cdot n!$ with the first few values $(1,-1,2,-6,24,-120,720,-5040,\ldots)$ (A133942) and that the sequence of the row sums is $S_1(n) = \sum_{k=0}^{n} U(n,k) = \frac{1+(-1)^n}{2} \cdot n!$ (A005359) with the first few values $(1,0,2,0,24,0,720,0,\ldots)$, while the sequence of the alternating row sums given by $S_2(n) = \sum_{k=0}^{n}(-1)^k U(n,k)$ with the first few values $(1,2,4,12,52,250,1608,10808,\ldots)$ is not in the OEIS.

By (2.12) the first few values of $b_n^{(r)}$ for fixed $r \geq 0$ are:

i) $r=0$: $b_n^{(0)} = (-1)^{n+1} n$ (A181983)

ii) $r=1$: $b_n^{(1)} = \frac{1}{4}(1-(2n+1)(-1)^n)$ (A001057, canonical enumeration of integers)

iii) $r=2$: $b_n^{(2)} = \frac{1}{4}(n+1)(1-(-1)^n)$ (A142150$(n+1)$, the nonnegative integers interleaved with 0's)

$n\backslash k$	0	1	2	3	4	5	6	7
0	1							
1	1	-1						
2	1	-1	2					
3	1	0	5	-6				
4	1	2	11	-14	24			
5	1	5	25	-5	94	-120		
6	1	9	55	75	304	-444	720	
7	1	14	112	350	1099	-364	3828	-5040

Table 4: Triangle $U(n,k)$ of the coefficients (in descending powers of r) of the polynomials $n! \cdot a_n^{(r)}$

iv) $r = 3$: $b_n^{(3)} = \frac{1}{16}(2n^2 + 6n + 3 - (2n+3)(-1)^n)$ (removing the first term this is the sequence A008805$(n-1)$, $n \geq 1$, triangular numbers repeated)

v) $r = 4$: $b_n^{(4)} = \frac{1}{48}(2n^3 + 12n^2 + 19n + 6 - 3(n+2)(-1)^n)$ (A006918)

vi) $r = 5$: $b_n^{(5)} = \frac{1}{192}(2n^4 + 20n^3 + 64n^2 + 70n + 15 - 3(2n+5)(-1)^n)$ (removing the first term this is the sequence A002624$(n-1)$, $n \geq 1$).

In particular, setting $r = 3$ and $n = 2m$, $m \in \mathbb{N}_0$, in (2.16) gives the sequence of the triangular numbers A000217

$$\sum_{k=0}^{2m} \binom{k+2}{2} (-1)^{k-1}(2m-k) = \frac{1}{2}m(m+1) = \binom{m+1}{2},$$

while for $n = 2m+1$, $m \in \mathbb{N}_0$, we also get the sequence of the triangular numbers A000217$(m+1)$

$$\sum_{k=0}^{2m+1} \binom{k+2}{2} (-1)^{k-1}(2m+1-k) = \frac{1}{2}(m+1)(m+2) = \binom{m+2}{2}.$$

On the other hand, again by (2.12) the first few sequences of $b_n^{(r)}$ for fixed $n \geq 0$ are:

i) $n = 0$: $b_0^{(r)} = 0$, (A000004, the zero sequence)

ii) $n = 1$: $b_1^{(r)} = 1$, (A000012, the all 1's sequence)

iii) $n = 2$: $b_2^{(r)} = r - 2$, (A023444)

$n\backslash k$	0	1	2	3	4	5	6
0	0						
1	1						
2	1	-2					
3	1	-3	6				
4	1	-3	14	-24			
5	1	-2	23	-70	120		
6	1	0	35	-120	444	-720	
7	1	3	55	-135	1024	-3108	5040

Table 5: Triangle $V(n,k)$ of the coefficients (in descending powers of r) of the polynomials $(n-1)! \cdot b_n^{(r)}$.

iv) $n = 3$: $b_3^{(r)} = \frac{1}{2}(r^2 - 3r + 6)$, (after removing the first term this is the sequence A152948)

v) $n = 4$: $b_4^{(r)} = \frac{1}{6}(r^3 - 3r^2 + 14r - 24)$, (this sequence is not in the OEIS)

vi) $n = 5$: $b_5^{(r)} = \frac{1}{24}(r^4 - 2r^3 + 23r^2 - 70r + 120)$, (this sequence is not in the OEIS).

Remark 2.7. A look at the polynomials $(n-1)! \cdot b_n^{(r)}$ (for fixed $n \geq 0$) shows that their coefficients $V(n,k)$, $n, k \in \mathbb{N}_0$, in descending powers of r are given by the triangle as shown in Table 5. Note also that $V(n, n-1) = (-1)^{n+1} \cdot n!$, $n \geq 1$, (sequence A155456(n+2)) with the first few values $(1, -2, 6, -24, 120, -720, 5040, \ldots)$ and that the sequence of the row sums is $T_1(n) = \sum_{k=0}^{n-1} V(n,k) = (-1)^{n+1} n! \cdot \frac{2n+3}{4}$, $n \geq 1$, (after removing the first term, the unsigned sequence $|T_1(n)|$ is A052558) with the first few values $(0, 1, -1, 4, -12, 72, -360, 2880, \ldots)$, while the sequence of the alternating row sums given by $T_2(n) = \sum_{k=0}^{n-1}(-1)^k V(n,k)$ with the first few values $(0, 1, 3, 10, 42, 216, 1320, 9366, \ldots)$ is not in the OEIS.

3 Generating functions and two special subsequences

We now determine the ordinary generating functions for $(a_n^{(r)})_{n\in\mathbb{N}_0}$ and $(b_n^{(r)})_{n\in\mathbb{N}_0}$, denoted by $F_r(s)$ and $G_r(s)$, respectively. We recall that the ordinary generating function for the sequence $(f_n)_{n\in\mathbb{N}_0}$ is defined as the (formal) power series $\sum_{n=0}^{\infty} f_n s^n$.

Proposition 3.1. *The ordinary generating function for $a_n^{(r)}$ is given by*

$$F_r(s) = \frac{1}{1+s} \cdot \frac{1}{(1-s)^r}, \tag{3.1}$$

and that for $b_n^{(r)}$ is given by

$$G_r(s) = \frac{s}{(1+s)^2} \cdot \frac{1}{(1-s)^r} = \frac{s}{1+s} \cdot F_r(s). \tag{3.2}$$

Proof. By definition and by [2, Equation (7.21)], we have for all $r \geq 1$:

$$F_r(s) = \sum_{n=0}^{\infty} a_n^{(r)} s^n = \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} a_k^{(r-1)} \Big) s^n = \frac{1}{1-s} F_{r-1}(s).$$

The solution of this recurrence relation is given by $F_r(s) = F_0(s) \cdot \frac{1}{(1-s)^r}$. Since the generating function $F_0(s)$ for $a_n = (-1)^n$ is given by the geometric series $F_0(s) = \sum_{n=0}^{\infty} (-1)^n s^n = \frac{1}{1-(-s)} = \frac{1}{1+s}$, the assertion (3.1) is proved. Similarly, by definition and by [2, Equation (7.21)], we have for all $r \geq 1$:

$$G_r(s) = \sum_{n=0}^{\infty} b_n^{(r)} s^n = \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} b_k^{(r-1)} \Big) s^n = \frac{1}{1-s} G_{r-1}(s)$$

which has the solution $G_r(s) = G_0(s) \cdot \frac{1}{(1-s)^r}$. The generating function $G_0(s)$ for $b_n = (-1)^{n+1} n$ can be determined in the following way

$$\begin{aligned} G_0(s) &= \sum_{n=0}^{\infty} (-1)^{n+1} n s^n = \sum_{n=0}^{\infty} (-1)^{n+1} (n+1-1) s^n \\ &= \frac{1}{s} \sum_{n=0}^{\infty} (-1)^{n+1} (n+1) s^{n+1} - \sum_{n=0}^{\infty} (-1)^{n+1} s^n \\ &= \frac{1}{s} \sum_{n=0}^{\infty} (-1)^n n s^n + \sum_{n=0}^{\infty} (-1)^n s^n = \frac{1}{s} G_0(s) + \frac{1}{1+s}. \end{aligned}$$

Solving for $G_0(s)$, we get the formula (3.2). □

Finally, we consider the two subsequences $(d_n)_{n \in \mathbb{N}_0}$ and $(e_n)_{n \in \mathbb{N}_0}$, defined as $d_n := a_n^{(n)}$ and $e_n := b_n^{(n)}$, $n \geq 0$, which form the main diagonal of the arrays $(a_n^{(r)})$ and $(b_n^{(r)})$, respectively (see Table 2 and Table 3). We recall that $(C(n))_{n \in \mathbb{N}_0}$ is the sequence of the Catalan numbers, the sequence A000108, defined by $C(n) := \frac{1}{n+1} \binom{2n}{n}$, $n \geq 0$.

Proposition 3.2. *The sequences* $(d_n)_{n\in\mathbb{N}_0}$ *and* $(e_n)_{n\in\mathbb{N}_0}$ *satisfy the recurrences*

$$d_0 = 1, \quad 2d_{n+1} = -d_n + (3n+1)C(n), \quad n \geq 0, \tag{3.3}$$

and

$$e_0 = 0, \quad e_1 = 1, \quad 4e_{n+1} + 4e_n + e_{n-1} = (9n-5)C(n-1), \quad n \geq 1. \tag{3.4}$$

The solutions are $d_n = \sum_{k=0}^{n} \binom{n+k-1}{k}(-1)^{n-k}$ *and* $e_n = \sum_{k=0}^{n} \binom{n+k-1}{k}(-1)^{n-k+1}(n-k)$, $n \geq 0$.

Proof. By definition, we have $d_0 = a_0^{(0)} = 1$. By (2.7) for $r = n+1$ it follows that

$$d_{n+1} = a_{n+1}^{(n+1)} = -a_n^{(n+1)} + \binom{2n+1}{n+1}. \tag{3.5}$$

Furthermore, by (2.9) for $r = n$, we have

$$2a_n^{(n+1)} = a_n^{(n)} + \binom{2n}{n} = d_n + \binom{2n}{n}.$$

Hence, substituting this equation into (3.5) multiplied by 2, we get

$$2d_{n+1} = -2a_n^{(n+1)} + 2\binom{2n+1}{n+1} = -d_n - \binom{2n}{n} + 2\binom{2n+1}{n},$$

and this is the recurrence (3.3), since $2\binom{2n+1}{n} - \binom{2n}{n} = \binom{2n}{n}(2\frac{2n+1}{n+1} - 1) = \frac{3n+1}{n+1}\binom{2n}{n} = (3n+1)C(n)$. The solution of the recurrence (3.3) is given by (2.3) for $r = n$.

By definition, we have $e_0 = b_0^{(0)} = 0$ and $e_1 = b_1^{(1)} = 1$. For $r = n+1$, we get from (2.8)

$$e_{n+1} = b_{n+1}^{(n+1)} = -2b_n^{(n+1)} - b_{n-1}^{(n+1)} + \binom{2n}{n}, \quad n \geq 1. \tag{3.6}$$

For $r = n$, we get from (2.11)

$$4b_n^{(n+1)} = 4b_n^{(n)} - b_n^{(n-1)} + \binom{2n-1}{n-1} = 4e_n - b_n^{(n-1)} + \frac{1}{2}\binom{2n}{n}, \quad n \geq 1, \tag{3.7}$$

and for $r = n$ and $n-1$ instead of n

$$4b_{n-1}^{(n+1)} = 4b_{n-1}^{(n)} - b_{n-1}^{(n-1)} + \binom{2n-2}{n-2} = 4b_{n-1}^{(n)} - e_{n-1} + \binom{2n-2}{n}, \quad n \geq 1. \tag{3.8}$$

Furthermore, by definition of the hypersequence, we have

$$e_n = b_n^{(n)} = b_{n-1}^{(n)} + b_n^{(n-1)}, \quad n \geq 1, \tag{3.9}$$

and

$$b_n^{(n+1)} = b_{n-1}^{(n+1)} + b_n^{(n)} = b_{n-1}^{(n+1)} + e_n, \quad n \geq 1. \tag{3.10}$$

Setting $b_{n-1}^{(n)} = \alpha$, $b_{n-1}^{(n+1)} = \beta$, $b_n^{(n-1)} = \gamma$, and $b_n^{(n+1)} = \delta$, we obtain from (3.6), (3.7), (3.8), (3.9) and (3.10) the following linear system consisting of 5 equations for the 4 unknowns α, β, γ, and δ.

$$\begin{aligned}
e_{n+1} &= -2\delta - \beta + \binom{2n}{n} \\
4\delta &= 4e_n - \gamma + \frac{1}{2}\binom{2n}{n} \\
4\beta &= 4\alpha - e_{n-1} + \binom{2n-2}{n} \\
e_n &= \alpha + \gamma \\
\delta &= \beta + e_n.
\end{aligned}$$

After algebraic elimination of the values α, β, γ, δ, the equation $4e_{n+1} + 4e_n + e_{n-1} = 2\binom{2n}{n} + \binom{2n-2}{n}$ remains, which is (3.4), since $2\binom{2n}{n} + \binom{2n-2}{n} = \frac{(2n-2)!}{(n-1)!(n-1)!} \cdot \left(2\frac{(2n-1)2n}{n \cdot n} + \frac{n-1}{n}\right) = \binom{2(n-1)}{n-1}\frac{9n-5}{n} = (9n-5)C(n-1)$, $n \geq 1$. The solution of the recurrence (3.4) is given by (2.4) for $r = n$. $\square$

The sequence $(d_n)_{n\in\mathbb{N}_0}$ with the first few values $(1, 0, 2, 6, 22, 80, 296, 1106, \ldots)$ is the sequence A072547, while the sequence $(e_n)_{n\in\mathbb{N}_0}$ with the first few values $(0, 1, 0, 3, 8, 30, 108, 399, \ldots)$ is not in the OEIS.

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