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Intrinsic geometry of cyclic polygons via new Brahmagupta's formula revisited

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Abstract

In this paper, we give full details for an intrinsic approach, using the author's New Brahmagupta formulas, to the computation of Heron polynomials for cyclic polygons (up to $n = 8$). A less complete account was already given in [20] (and used by S. Moritsugu, see ref. [27]) following the author's talk at the International Congress of Mathematicians in Hyderabad, India, in 2010. We also mention a new approach by multivariate discriminants based on the fact that the cyclic polygons are critical points of the area functional.

1 Introduction

Finding explicit equations for the area or circumradius of polygons inscribed in a circle in terms of side lengths is a classical subject (cf. [1]). For triangle / cyclic quadrilaterals, we have the famous Heron / Brahmagupta formulae. In 1994. D. P. Robbins found a minimal area equation for cyclic pentagons/hexagons by a method of undetermined coefficients (cf. [5]). This method could hardly be used for heptagons due to computational complexity (143307 equations).

In [8], by using covariants of binary quintics, a concise minimal heptagon/octagon area equation was obtained as a quotient of two resultants, which in expanded form has almost one million terms. It is not clear if this approach could be effectively used for cyclic polygons with nine or more sides.

In [15] and [28], by using the Wiener-Hopf factorization approach, we have obtained a very explicit minimal heptagon/octagon circumradius equation

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(only 13 pages formula) in Pellian form (= a combination of two squares of smaller polynomials whose coefficients have at most four digits). A non-minimal area equation is also obtainable by this method. Both methods are somehow external.

But, based on our new intermediate Brahmagupta formulas (2.6) and (2.7), we have succeeded in finding a direct intrinsic proof of the Robbins formulas for the area (and also for circumradius and area times circumradius) of cyclic hexagons.

Earlier, an intricate direct elimination of diagonals for cyclic hexagons was painful (see the footnote on the page 117) (the case of a pentagon was much easier, cf. [21]).

We also get a simple(st) system of equations ($EQ1$, $EQ2$, $EQ3$ on page 121) for the area (and area times circumradius) of cyclic octagons.

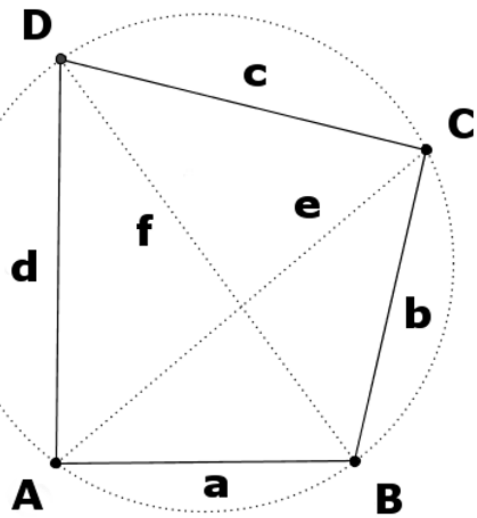
It seems remarkable that our approach, with the help of Gröbner basis techniques, leads to minimal equations (for any concrete instances we have tested), which is not the case with the iterated resultants approach.

Inspired by our observation on page 119 at the end we present a new method of multivariate discriminants, for finding area equation for cyclic octagons, of a master equation by using the result (cf. [16]) that cyclic polygons are critical points of the area functional.

For reader convenience we recommend a somewhat older survey [9] by I. Pak and references [22]– [27] by S. Moritsugu who used our reference [20] in [27].

We hope that our method of dissecting cyclic polygons into cyclic quadrilaterals is concordant with well known Grothendieck's well-known reconstruction principle.

2 Cyclic quadrilaterals



We first recall some basic formulas for cyclic quadrilaterals $ABCD$ with sides and diagonals of lengths $a = |AB|$, $b = |BC|$, $c = |CD|$, $d = |DA|$ and $e = |AC|$, $f = |BD|$ whose vertices lie on a circle of radius R.

- **Ptolemy's relation** (convex case):

$$ef = ac + bd \tag{2.1}$$

- **Dual Ptolemy's relation**:

$$(ab + cd)e = (ad + bc)f \quad (= 4SR) \tag{2.1'}$$

- **Diagonal equation**:

$$(ab + cd)\mathbf{e^2} = (ac + bd)(ad + bc) \tag{2.2}$$

- **Area equation** (Brahmagupta's formula, 625. AD):

$$16S^2 = 2(a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2)-a^4-b^4-c^4-d^4+8abcd \tag{2.3}$$

  which, in a more popular form, reads as

$$16S^2 = (-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d) \tag{2.3'}$$

- **Circumradius equation** (Parameshavara's formula, 1400. AD):

$$R^2 = \frac{(ab+cd)(ac+bd)(ad+bc)}{(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d)} \tag{2.4}$$

- **Area times circumradius equation**:
  Let $Z = 4SR$, then

$$Z^2 = (ab + cd)(ac + bd)(ad + bc) \tag{2.5}$$

which in a case of a triangle ($d = 0$) reduces to the well known relation

$$4SR = abc. \tag{2.5'}$$

For the reader's convenience, one of the simplest methods for obtaining Brahmagupta's formula makes use of trigonometry: for the interior angles at B and D we have $B+D = 180°$, implying $\cos D = -\cos B$, $\sin D = \sin B$. By the Law of Cosine, we obtain $2(ab + cd)\cos B = a^2 + b^2 - c^2 - d^2$. For area $S = \frac{1}{2}ab\sin B + \frac{1}{2}cd\sin D = \frac{1}{2}(ab + cd)\sin B$. Hence $16S^2 = (2ab+2cd)^2-(a^2+b^2-c^2-d^2)^2 = (2s-2a)(2s-2b)(2s-2c)(2s-2d)$, where $2s = a+b+c+d$. This completes the classical proof of Brahmagupta's formula.

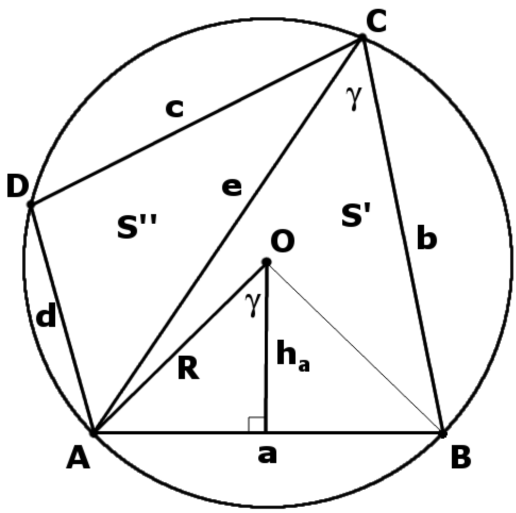
Our main contribution is the following discovery: the Key Lemma and a new (atomic) Brahmagupta's formula.
This lemma will be crucial in all our subsequent calculations concerning the elimination of diagonals in cyclic polygons.

Key Lemma: (Intermediate Brahmagupta's formula)
In any convex cyclic quadrilateral, we have

$$\boxed{8Sh_a = 2bcd + (b^2 + c^2 + d^2 - a^2)a} \tag{2.6}$$

where h_a denotes the height (positive or negative) of the center of the circumcircle with respect to the side AB.

In the case of a nonconvex quadrilaterals, we can formally obtain all the relations by simply allowing side lengths to be negative (e.g. by replacing a with $-a$).



Proof of the Key Lemma. $S = S' + S'' \Rightarrow$
$4RS = 4RS' + 4RS'' = abe + cde$ (Dual Ptolemy's relation)

By Law of Cosine, dual Ptolemy's and Diagonal equation for $h_a \geq 0$ we have: $h_a = R\cos\gamma = R\dfrac{b^2+e^2-a^2}{2be} = \dfrac{(ab+cd)(b^2+e^2-a^2)}{8Sb}$
$$= \frac{2bcd + (b^2+c^2+d^2)a - a^3}{8S}.$$
(Case $h_a < 0$ is similar.) ■
Let $S_a = \frac{ah_a}{2}$ be the signed area of the characteristic triangle $\triangle OAB$ determined by the side AB (of length a) and circumcenter O of a cyclic quadrilateral $ABCD$. Then we get

Corollary 2.1. *(New Brahmagupta's formulas)*

$$\boxed{16SS_a = a^2(b^2+c^2+d^2-a^2) + 2abcd} \tag{2.7}$$

and three more formulas, by cyclically permuting a, b, c and d.

Note that by adding all four such formulas we get the original Brahmagupta's formula because

$$S = S_a + S_b + S_c + S_d.$$

For general quadrilaterals in a plane, we have:

- **Bretschneider's formula ([2]) or Staudt's formula (1842)**:

$$16S^2 = 4e^2f^2 - (a^2-b^2+c^2-d^2)^2. \tag{2.8}$$

For cyclic quadrilaterals, in view of (2.1), it gives another form of (2.3):

$$16S^2 = 4(ac+bd)^2 - (a^2 - b^2 + c^2 - d^2)^2. \qquad (2.3")$$

The formula (2.8) is the simplest formula for the area of the quadrilateral in terms of its sides and diagonals. But there are infinitely many other ways to do so, since these 6 quantities satisfy Euler's four-point relation

$$\begin{aligned} & e^2 f^2 (a^2 + b^2 + c^2 + d^2 - e^2 - f^2) = \\ & = e^2(a^2 - b^2)(d^2 - c^2) + f^2(a^2 - d^2)(b^2 - c^2) + \\ & \quad + (a^2 - b^2 + c^2 - d^2)(a^2c^2 - b^2d^2) \end{aligned} \qquad (2.9)$$

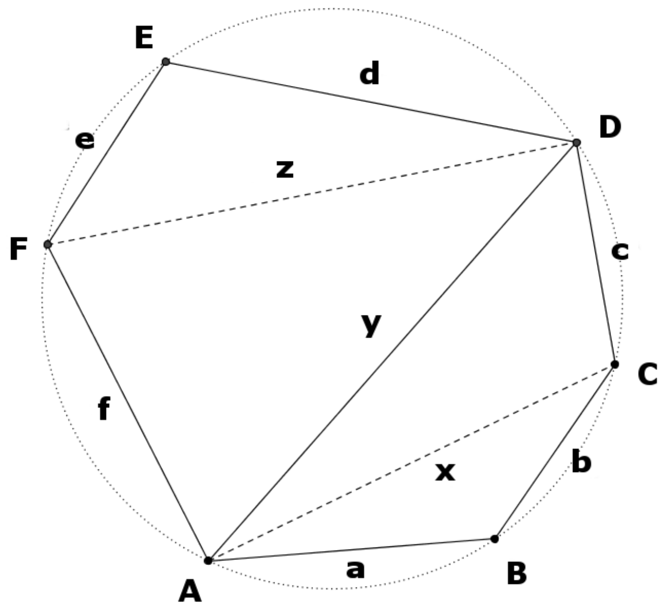
This is only a quadratic equation with respect to a square of each parameter. The Euler's four point relation follows from the Cayley–Menger determinant for the volume V of a tetrahedron with edges of lengths a, b, c, d, e, f if we set $V = 0$.

Remark 2.2. In a solution of a problem by J.W.L.Glaisher: With four given straight lines to form a quadrilateral inscribable in a circle, A.Cayley (in 1874.) observed the following identity, equivalent to (2.9):

$$\begin{aligned} & [(a^2 + b^2 + c^2 + d^2 - e^2 - f^2)(ef + ac + bd) - 2(ad + bc)(ab + cd) \cdot] \\ & \cdot (ef - ac - bd) = [(ab + cd)e - (bc + ad)f]^2 \end{aligned} \qquad (2.9')$$

which directly shows that Ptolemy's relation (2.1) implies the dual Ptolemy's relation (2.1').

3 Cyclic hexagons



Cyclic hexagon $ABCDEF$ inscribed in a circle of radius R, with side lengths $a = |AB|$, $b = |BC|$, $c = |CD|$, $d = |DE|$, $e = |EF|$, $f = |FA|$, $\boldsymbol{y}$ = **main** diagonal, $\boldsymbol{x}$,$\boldsymbol{z}$ = **small** diagonals.

- **Main diagonal equation**

Let $y = |AD|$ denote the length of the main diagonal of the cyclic hexagon $ABCDEF$. Then we may think of the hexagon $ABCDEF$ as made up of two quadrilaterals with a common side AD, both having the same circumradius R. Thus using the formula (2.4) twice we get equality

$$
\begin{aligned}
&(R^2 =) \\
&\frac{(de+fy)(df+ey)(ef+dy)}{(-d+e+f+y)(d-e+f+y)(d+e-f+y)(d+e+f-y)} = \\
&= \frac{(ab+cy)(ac+by)(bc+ay)}{(-a+b+c+y)(a-b+c+y)(a+b-c+y)(a+b+c-y)}
\end{aligned}
\tag{3.1}
$$

leading to a 7–th degree equation

$$\boxed{(def-abc)y^7+\cdots=0}$$

for the length of the main diagonal y.

With substitutions

$$u = a^2 + b^2 + c^2, \quad v = a^2b^2 + a^2c^2 + b^2c^2, \quad w = abc \tag{3.2}$$

$$U = d^2 + e^2 + f^2, \quad V = d^2e^2 + d^2f^2 + e^2f^2, \quad W = def \tag{3.2'}$$

we can express the area S' (resp. S'') of the quadrilateral $ABCD$ (resp. $ADEF$) as follows

$$16S'^2 = 4v - u^2 + 8wy + 2uy^2 - y^4, \quad 16S''^2 = 4V - U^2 + 8Wy + 2Uy^2 - y^4 \tag{3.3}$$

Then (3.1) becomes equivalent to

$$P_6^{\text{main diag.}} \equiv -S''^2\left(wy^3 + vy^2 + uwy + w^2\right) + \\ + S'^2\left(Wy^3 + Vy^2 + UWy + W^2\right) = 0 \tag{3.1'}$$

$$\text{(i.e.} \boxed{(w - W)y^7 + (v - V)y^6 + \cdots + (4v - u^2)W^2 - (4V - U^2)w^2 = 0)} \tag{3.1''}$$

By letting $f = 0$, we obtain the diagonal equation for a cyclic pentagon $ABCDE$:

$$\boxed{\begin{array}{l} P_5^{\text{diag.}} \equiv \\ abc\ y^7 + (a^2b^2 + a^2c^2 + b^2c^2 - d^2e^2)y^6 + \cdots + a^2b^2c^2(d^2 - e^2) = 0 \end{array}}$$

(cf. Bowman [4]).

- **Small diagonal equation**

Let $x = |AC|$ denote the length of a "small" diagonal in the cyclic hexagon $ABCDEF$. By (2.2) we obtain the equation

$$(ab + cy)x^2 = (ac + by)(bc + ay)$$

by which we can eliminate y in our main diagonal equation (3.1"). This gives our small diagonal equation, which has degree 7 in x^2:

$$\begin{array}{l} P_6^{\text{small diag.}} \equiv \\ (abc - def)(abd - cef)(abe - cdf)(abf - cde)x^{14} + (\ldots)x^{12} + \cdots + \\ +(a^2 - b^2)^4(acd - bef)(ace - bdf)(acf - bde)(ade - bcf) \\ (adf - bce)(aef - bcd) = 0 \end{array} \tag{3.4}$$

By letting $f = 0$ we obtain

$$P_6^{\text{small diag.}}\Big|_{f=0} = a^3 b^3 P_5^{\text{diag.}} \left(P_5^{\text{diag.}}\right)^* \tag{3.4'}$$

where $P_5^{\text{diag.}} \equiv cde\ x^7 + \cdots = 0$ and $\left(P_5^{\text{diag.}}\right)^*$ is obtained by changing sign of an odd number of side lengths c, d, e.

- **Area equation**: Naive approach

A naive approach to get the area equation of cyclic hexagon would be to write the area S of our hexagon as

$$S = S' + S'' \tag{3.5}$$

Then by rationalizing the equation (3.5) we obtain an equation of degree 4 in y:

$$(S^2 + S'^2 - S''^2)^2 - 4S^2 S'^2 = 0 \tag{3.6}$$

where S'^2 and S''^2 are given by Brahmagupta's formula (3.3). More explicitly, in terms of the squared area $A = (4S)^2$ we have

$$\begin{aligned} Q \equiv & \\ & (A + 4(v - V) + U^2 - u^2 + 8(w - W)y + 2(u - U)y^2)^2 - \\ & - 4A(4v - u^2 + 8wy + 2uy^2 - y^4) = 0 \end{aligned} \tag{3.6'}$$

By computing the resultant of this equation and the main diagonal equation (3.1') w.r.t. y we obtain a degree 14 polynomial in A.

$$\text{Resultant}\left(Eq(3.6'),\ P_6^{\text{main diag.}},\ y\right) = F_1 F_2$$

both of whose factors have degree 7 in A:

$$\begin{aligned} F_1 &= (w - W)^2 A^7 + \cdots \\ F_2 &= A^7 + (7u^2 + 7U^2 - 10uU - 24v - 24V)A^6 + \cdots \end{aligned}$$

The true equation (obtained first by Robbins in 1994. by undetermined coefficients method) is given by F_2 (it has 2042 monomials), and the extraneous factor F_1 (which has 8930 monomials) is 4 time bigger[1].

- **Area equation**: new approach leading to an intrinsic proof.

[1]The computation with MAPLE 9.5 on a PC with 2GHz and 2GB RAM took $\approx$ 300 hours (in year 2004). Nowadays with MAPLE 12 on a 64–bit PC with 8GB it takes $\approx$ 3 hours.

The complications with the extraneous factor in the previous proof were probably caused by using squaring operation twice in order to get the equation (3.6) (or (3.6')). So we are searching a simpler equation relating the area S and the main diagonal. After a long struggle we obtained an extraordinary simple relationship given in the following

Key Lemma. The area S of the cyclic hexagon $ABCDEF$ and areas S' and S'' of the cyclic quadrilaterals $ABCD$ and $ADEF$ obtained by subdivision with the main diagonal of length $y = |AD|$ satisfy the following relations:

a) $(y^3 - (a^2 + b^2 + c^2)y - 2abc)S'' + (y^3 - (d^2 + e^2 + f^2)y - 2def)S' = 0$

b) $(y^3-(a^2+b^2+c^2)y-2abc)S+((a^2+b^2+c^2-d^2-e^2-f^2)y+2(abc-def))S' = 0$

Proof. a) Let $x = |AC|$, $y = |AD|$, $z = |DF|$. Let S_1', S_2' S_1'' and S_2'' be the areas of triangles ABC, ACD, ADF and DEF respectively. Then, by (2.5') we have $4S_1'R = abx$, $4S_2'R = cxy$, $4S_1''R = fyz$, $4S_2''R = dez$. So we have $4S'R = (ab + cy)x$, $4S''R = (fy + de)z$. This implies

$$\frac{S''}{S'} = \frac{fy + de}{ab + cy} \cdot \frac{z}{x}$$

The diagonal equation for the main diagonal $y = |AD|$ in the middle quadrilateral $ACDF$: $(cx + fz)y^2 = (cf + xz)(fx + cz)$ can be rewritten as

$$cx(y^2 - f^2 - z^2) = fz(-y^2 + c^2 + x^2)$$

Now we have

$$\frac{S''}{S'} = \frac{fy + de}{ab + cy} \cdot \frac{y^2 - f^2 - z^2}{x^2 + c^2 - y^2} \cdot \frac{c}{f} = \frac{c}{f} \frac{(fy + de)(y^2 - f^2) - (fy + de)z^2}{(ab + cy)(c^2 - y^2) + (ab + cy)x^2}$$

Finally we use the diagonal equations for small diagonals x and z in respective quadrilaterals

$$(ab + cy)x^2 = (ac + by)(bc + ay), \quad (fy + de)z^2 = (df + ey)(ef + dy)$$

and by simplifying we get

$$\frac{S''}{S'} = \frac{y^3 - (d^2 + e^2 + f^2)y - 2def}{2abc + (a^2 + b^2 + c^2)y - y^3}$$

b) follows from a) by substituting $S'' = S - S'$. $\square$

By writing the equation b) in Key Lemma with shorthand notations (3.2) and (3.2')

$$(y^3 - uy - 2w)S + ((u - U)y + 2(w - W))S' = 0$$

and multiplying it by $2S$, $2S'$ respectively and using the relation

$$2SS' = S^2 + S'^2 - S''^2$$

obtained from (3.5) by squaring, we obtain the following

KEY EQUATIONS:

$$\boxed{Q_1 := 2(y^3 - uy - 2w)S^2 + ((u - U)y + 2(w - W))(S^2 + S'^2 - S''^2) = 0}$$

$$\boxed{Q_2 := (y^3 - uy - 2w)(S^2 + S'^2 - S''^2) + 2((u - U)y + 2(w - W))S'^2 = 0}$$

where S'^2 and S''^2 are given by Brahmagupta's formulas (3.3).

MAIN THEOREM. The resultant of the Key Equations with respect to y gives the minimal degree 7 equation for the squared area $A = (4S)^2$ of cyclic hexagon.

Proof. The minimal polynomial

$$\alpha_6 = \text{Resultant}(Q_1, Q_2, y)/C = A^7 + (7(u^2+U^2) - 10uU - 24(v+V))A^6 + \cdots$$

where $C = 4\left[4(W - w)^3 + (u - U)^3(wU - uW)\right]$.
Remark. Observe that $16Q_1 = [2A + 2(u - U)^2]y^3 + \cdots$
Similarly the polynomial Q in equation (3.6') has the form

$$Q = \left[4A + 2(u - U)^2\right] y^4 + \cdots$$

If we define

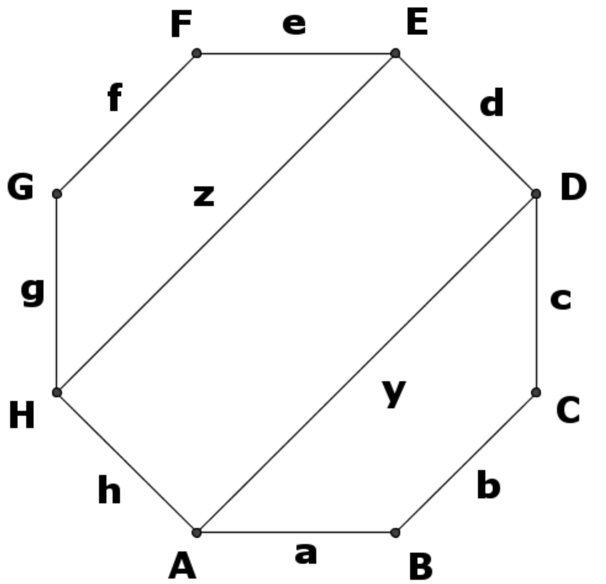
$$\begin{aligned} Q_3 &:= Q - 2 \cdot 16Q_1 \\ &= 4(-uy^2 - 6wy - 4v + u^2)A + (4(v + w - V - W) + U^2 - u^2 + A)\cdot \\ &\quad \cdot (A + 4(v - V) + U^2 - u^2 + 8(w - W)y + 2(u - U)y^2) \end{aligned}$$

then we also get

$$\alpha_6 = \text{Resultant}(Q_3, 16Q_1, y)/(-8A^2)$$

New Observation! By using the fact that cyclic polygons are critical points of area (c.f. [16]) we can obtain New Theorem which uses the theory of discriminants $:= discrim(Q, y)/(2^1 4A^2) = A^7 + \cdots$. Where Q is given in (3.6').

4 Area equations of cyclic octagons and (heptagons)



We trisect cyclic octagon $ABCDEFGH$, by two diagonals AD and EH into three quadrilaterals $ABCD$, $ADEH$ and $EFGH$ whose areas we denote by S_1, S_2 and S_3 respectively. The area S of $ABCDEFGH$ is then equal to

$$S = S_1 + S_2 + S_3 \tag{4.1}$$

By Key Lemma a) applied to hexagons $ABCDEH$ and $ADEFGH$ we obtain the following equations:

$$(2jz + (i + z^2)y - y^3)S_1 + (2w + uy - y^3)S_2 = 0 \tag{4.2}$$
$$(2jy + (i + y^2)z - z^3)S_3 + (2W + Uz - z^3)S_2 = 0 \tag{4.3}$$

where we have used the following abbreviations:

$$\begin{aligned}
&y = |AD|, z = |EH| \\
&u = a^2 + b^2 + c^2, v = a^2b^2 + a^2c^2 + b^2c^2, w = abc \\
&U = e^2 + f^2 + g^2, V = e^2f^2 + e^2g^2 + f^2g^2, w = efg \\
&i = d^2 + h^2, j = dh
\end{aligned} \tag{4.4}$$

Furthermore the Brahmagupta formulas for the 16 times squared areas $A_i =$

$16S_i^2$, $i = 1, 2, 3$ can be written now as follows:

$$A_1 = 4v - u^2 + 8wy + 2uy^2 - y^4 \tag{4.5}$$
$$A_2 = 4(j + yz)^2 - (y^2 + z^2 - i)^2 \tag{4.6}$$
$$A_3 = 4V - U^2 + 8Wz + 2Uz^2 - z^4 \tag{4.7}$$

(For A_1, A_3 cf. (3.3), for A_2 cf. (2.3") from Preliminaries!)
By equating the circumradius formulas for cyclic quadrilaterals $ABCD$ and $ADEH$ (resp. $ABCD$ and $EFGH$) we obtain two equations:

$$\boxed{\begin{aligned} &EQ1 := (4v - u^2 + 8wy + 2uy^2 - y^4)(jzy^3 + (iz^2 + j^2)y^2 + (i + z^2)jzy+ \\ &+(jz)^2) - (4(j + yz)^2 - (y^2 + z^2 - i)^2)(wy^3 + vy^2 + uwy + w^2) = 0 \end{aligned}} \tag{4.8}$$

$$\boxed{\begin{aligned} EQ2 := (4v - u^2 + 8wy + 2uy^2 - y^4)(Wz^3 + Vz^2 + UWz + W^2)- \\ - (4V - U^2 + 8Wz + 2Uz^2 - z^4)(wy^3 + vy^2 + uwy + w^2) = 0 \end{aligned}} \tag{4.9}$$

Our next aim is to get one more equation (as simple as possible) relating the lengths y and z of diagonals and the squared area $A = 16S^2$ of our cyclic octagon. Here is a result of a many years long search:

Theorem 4.1. *(Fundamental equation involving area of cyclic octagons) Let $A = 16S^2$ be the squared area of any cyclic octagon. Then we have the following equation of degree 6 in y and z and linear in A:*

$$\boxed{EQ3 := \alpha\gamma(A + \eta) + 2(\alpha - \beta)(\delta - \gamma)A_2 = 0} \tag{4.10}$$

where

$$\begin{aligned} &\alpha = 2jz + iy + yz^2 - y^3, \quad \beta = 2w + uy - y^3 \\ &\gamma = 2jy + iz + y^2z - z^3, \quad \delta = 2W + Uz - z^3 \\ &\eta = u^2 + U^2 - i^2 - 4v - 4V + 4j^2 - 8wj - 8Wz + 8jyz+ \\ &\quad + 2(i - u)y^2 + 2(i - U)z^2 + 2y^2z^2 \end{aligned}$$

Proof. We start by squaring the equation (4.1)

$$S^2 = S_1^2 + S_2^2 + S_3^2 + 2S_1S_2 + 2S_1S_3 + 2S_2S_3 \tag{4.11}$$

Solving (4.2) for S_1 and (4.3) for S_2 yields:

$$S_1 = -\frac{\beta}{\alpha}S_2, \quad S_3 = -\frac{\delta}{\gamma}S_2 \tag{4.12}$$

Then we substitute these only into the mixed terms of (4.11). This gives:

$$S^2 = S_1^2 + S_2^2 + S_3^2 + 2\left(-\frac{\beta}{\alpha} + \frac{\beta\delta}{\alpha\gamma} - \frac{\delta}{\gamma}\right) S_2^2$$

By multiplying the last equation by 16 and using that $A_i = 16S_i^2$, $A = 16S^2$ we obtain

$$\alpha\gamma(A - A_1 + A_2 - A_3) + 2(\alpha - \beta)(\delta - \gamma)A_2 = 0$$

and set

$$\eta = -A_1 + A_2 - A_3$$

and the result follows by (4.5), (4.6) and (4.7). □

Remark 4.2. By using Gröbner basis for $\{EQ1, EQ2, EQ3\}$ we get minimal equation α_7 (α_8) for squared area ($A = 16\,Area^2$) of cyclic heptagons (octagons) in concrete instances very fast.

Remark 4.3. Maley M.F., Robins D.P. and Roskies J. ([8]) obtained explicit formulas for α_7 and α_8 in terms of elementary symmetric functions of sides lengths squared.

$$\alpha_7 = \frac{2^{101}5^5 Res(\widetilde{F}, \widetilde{G}, u_3)}{u_2^4 Res(\widetilde{F}_1, \widetilde{F}_2, u_3)}$$

Half a year later we have fully expanded α_7 which has 955641 terms with up to 40-digits coefficients (approx. 5000 pages).

Remark 4.4. For ζ_7, the $Z(= 4SR)$-polynomial, by a similar method, we obtained explicit formula with 31590 terms with up to 11 digits coefficients.

Remark 4.5. For $\rho_7 = R^2$-equation of cyclic heptagon, by a different technique, we obtained a 15 pages output in a condensed (Pellian) form – a quadratic form of two smaller polynomials whose coefficients have up to 4 digits coefficients in terms of new quantities (which are certain linear combinations of elementary symmetric functions of side lengths squared) published explicitly in [28].

5 Area equations for cyclic octagons by using bivariate discriminants

We start with a cyclic octagon $ABCDEFGH$, trisected by two diagonals AD and EH into three quadrilaterals $ABCD$, $ADEH$ and $EFGH$ whose areas are S_1, S_2 and S_3 respectively. The area S of $ABCDEFGH$ is then

$$S = S_1 + S_2 + S_3\,. \tag{5.1}$$

For the squared areas $A_i = 16S_i^2$, $i = 1,2,3$ we have the formulas (4.5 ... 4.7) relying on the abbreviations (4.4).
The rationalized form of (4.1) can be written compactly as follows:

$$\Big[(A - A_1 - A_2 - A_3)^2 - 4\,(A_1A_2 + A_1A_3 + A_2A_3)\Big]^2 - 64AA_1A_2A_3 = 0\,. \tag{$\bigstar$}$$

(This is in fact a general Brahmagupta polynomial evaluated at $a_i^2 = A_i$, $i = 1,2,3$, $a_4^2 = A$).
By inserting

$A_1 = 4v - u^2 + 8wy + 2uy^2 - y^4$ from (4.5),

$A_2 = 4(j + yz)^2 - (y^2 + z^2 - i)^2$ from (4.6) and

$A_3 = 4V - U^2 + 8Wz + 2Uz^2 - z^4$ from (4.7)

with

$y = |AD|, z = |EH|$

$u = a^2 + b^2 + c^2, v = a^2b^2 + a^2c^2 + b^2c^2, w = abc$

$U = e^2 + f^2 + g^2, V = e^2f^2 + e^2g^2 + f^2g^2, w = efg$

$i = d^2 + h^2, j = dh$

from (4.4) we obtain a master equation

$$\mathcal{M}(A, u, v, w, U, V, W, i, j, y, z) = 0\,.$$

Then the area equation is the discriminant of this master equation!
We conjecture similar results for arbitrary even sided cyclic polygons.

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