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Normal edge-colorings and superpositions: an overview

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Abstract

A normal 5-edge-coloring of a cubic graph is a coloring such that for every edge, the number of distinct colors incident to its end-vertices is 3 or 5 (and not 4). The well-known Petersen Coloring Conjecture is equivalent to the statement that every bridgeless cubic graph has a normal 5-edge-coloring. All 3-edge-colorings of a cubic graph are obviously normal, so in order to establish the conjecture, it is sufficient to consider only snarks. The most general known method for constructing snarks is superposition. In this paper, we give an overview of our results on the normal 5-edge-colorings of superpositioned snarks. A family of superpositioned snarks considered here is obtained from a snark G by superpositioning vertices and edges along a cycle C of G by two specific supervertices and by superedges of the form $H_{x,y}$, where H is any snark and x, y a pair of non-adjacent vertices in H. We assume that a snark G has a normal 5-edge-coloring σ and we extend σ to a superpositioned snark $\tilde{G}$. Our consideration starts with superpositions by the Petersen graph P_{10}, where we encounter problems with superpositions along odd cycles. We provide an example of a superposition by P_{10} along an odd cycle C in which σ cannot be extended to a superposition. This does not contradict the Petersen coloring conjecture, since the superposition does have a normal 5-edge-coloring, but not such that it is an extension of σ. We generalize our approach to superpositions by any superedge $H_{x,y}$, where $d(x, y) \geq 3$. For such superpositions, we give two sufficient conditions under which σ can be extended to a superposition. These conditions are applied to superpositions by Hypohamiltonian snarks

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and by Flower snarks, showing thus that some of the former and all of the latter have a normal 5-edge-colorings. Since the Petersen Coloring Conjecture implies some other well-known classical conjectures like the Ford-Fulkerson Conjecture, these results immediately yield some known results on this conjecture.

Keywords: normal edge-coloring; cubic graph; snark; superposition; Petersen Coloring Conjecture.
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1 Introduction

A *k-edge-coloring* of a graph G is a function $\sigma : E(G) \to \{1, \ldots, k\}$. If an edge-coloring assigns distinct colors to any two adjacent edges in G, it is said to be *proper*. Throughout the paper, we will omit the word 'proper' tacitly assuming properness unless explicitly stated otherwise. For any vertex $v \in V(G)$, the set of colors associated with the edges incident to v is denoted by $\sigma(v)$.

Definition 1.1. Consider a bridgeless cubic graph G, a proper edge-coloring σ, and an edge $uv \in E(G)$. The edge uv is defined as *poor* if $|\sigma(u) \cup \sigma(v)| = 3$, and as *rich* if $|\sigma(u) \cup \sigma(v)| = 5$.

An edge-coloring of a cubic graph G is said to be a *normal edge-coloring* if all edges of G are either poor or rich. This concept was first introduced by Jaeger in [10]. The *normal chromatic index* of G, written as $\chi'_N(G)$, is the minimum value of k for which a normal k-edge-coloring exists. Notably, $\chi'_N(G)$ is always at least 3, and it can never equal 4.
The Petersen Coloring Conjecture is one of the most prominent open problems in graph theory. This conjecture is particularly challenging to prove, as it has been shown to imply several other well-known conjectures, including the Berge-Fulkerson Conjecture and the (5,2)-cycle-cover Conjecture. Interestingly, the Petersen Coloring Conjecture can be reformulated in terms of normal edge-colorings, as noted in [10].

Conjecture 1.2. If G is a bridgeless cubic graph, then $\chi'_N(G) \le 5$.

It is evident that Conjecture 1.2 holds for every cubic graph G that admits a proper 3-edge-coloring, as such a coloring is a normal edge-coloring where all edges are poor. By Vizing's theorem, every cubic graph is either 3-edge-colorable or 4-edge-colorable. Therefore, to confirm Conjecture 1.2, it suffices to show that it applies to all bridgeless cubic graphs that are not 3-edge-colorable.

Superpositioning snarks. Cubic graphs that are not 3-edge-colorable are commonly referred to as snarks [5, 21]. To exclude trivial cases, the definition of a snark often includes additional conditions related to connectivity. However, these conditions are not crucial for the purposes of this paper. Hence, we adopt a broader definition, considering a *snark* to be any bridgeless cubic graph that is not 3-edge-colorable. Some families of snarks have already been shown to admit normal 5-edge-colorings; see, for instance, [4, 7].
The most general method currently known for generating new snarks from existing ones is the process of superposition [1, 3, 11, 12, 16]. Since this paper explores certain snarks created through superposition, we begin by introducing the method.

Definition 1.3. A *multipole* $M = (V, E, S)$ is defined by a set of vertices $V = V(M)$, a set of edges $E = E(M)$, and a set of semiedges $S = S(M)$. A semiedge is either incident to a single vertex or paired with another semiedge, forming what is known as an *isolated edge* within the multipole.

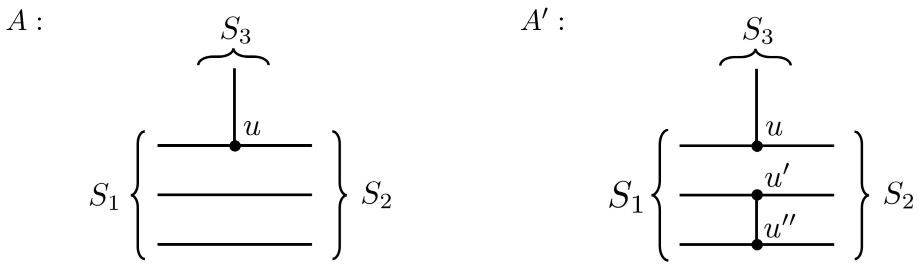

Figure 1: Supervertices A and A', with connectors S_1, S_2, and S_3, where S_1 is a 1-connector, and S_2 and S_3 are 3-connectors.

An example of a multipole can be found in Figure 1. For any vertex v in a multipole M, the *degree* $d_M(v)$ is defined as the total number of edges and semiedges in M that are incident to v. A multipole M is termed *cubic* if every vertex of M has degree 3. For instance, both multipoles depicted in Figure 1 are cubic. Throughout this paper, we focus exclusively on cubic multipoles.
Now, let us introduce some terminology related to the semiedges of a multipole M. A multipole M is referred to as a *k-pole* when the total number of semiedges, $|S(M)|$, equals k. If the set of semiedges S is divided into n subsets S_i such that $|S_i| = k_i$, the multipole is called a $(k_1, \dots, k_n)$*-pole* and is denoted as $M = (V, E, S_1, \dots, S_n)$. These subsets S_i are referred to as

the *connectors* of M. A connector S_i containing k_i semiedges is specifically called a k_i*-connector*.

Definition 1.4. A *supervertex* (respectively, a *superedge*) is defined as a cubic multipole with three (respectively, two) connectors.

We specifically define the supervertices A and A' as shown in Figure 1. These two supervertices will be the focus of our analysis throughout the paper. Next, we introduce a particular type of superedge relevant to our discussion. Let G be a snark, and let u and v be two non-adjacent vertices in G. The superedge $G_{u,v}$ is constructed by removing the vertices u and v from G, and replacing the three edges incident to u (and similarly for v) with three semiedges in $G_{u,v}$, which collectively form a connector.

Definition 1.5. A *proper superedge* is defined as either an isolated edge or a superedge $G_{u,v}$ where G is a snark.

While the definition of a proper superedge given in [11] is much broader, the simplified definition presented here is sufficient for the scope of this paper. In our work, we will also explore normal edge-colorings of multipoles, as the concept of normal edge-coloring extends naturally to these structures. Let us now formalize this notion.

For a multipole $M = (V, E, S)$, a *(proper) k-edge-coloring* is defined as a function $\sigma : E(M) \cup E(S) \to \{1, \dots, k\}$ such that no two edges or semiedges sharing the same color are incident to the same vertex. Furthermore, a *normal edge-coloring* of a multipole is a proper edge-coloring where every edge is either rich or poor. It is important to note that this definition places no restrictions on the coloring of semiedges.

For a cubic graph $G = (V, E)$, we define two functions: $\mathcal{V}$, which maps each vertex $v \in V$ to a supervertex $\mathcal{V}(v)$, and $\mathcal{E}$, which maps each edge $e \in E$ to a superedge $\mathcal{E}(e)$. A *superposition* $G(\mathcal{V}, \mathcal{E})$ is constructed under the following condition: semiedges of a connector in $\mathcal{V}(v)$ are matched with semiedges of a connector in $\mathcal{E}(e)$ if and only if e is incident to v in G. Naturally, this requires the connectors in $\mathcal{V}(v)$ and $\mathcal{E}(e)$ to have the same number of semiedges.

Observe that the resulting graph $G(\mathcal{V}, \mathcal{E})$ is again cubic. Such a superposition is said to be *proper* if every superedge $\mathcal{E}(e)$ is proper. Additionally, we assume that some vertices and edges of G may be superpositioned by themselves. Formally, such vertices are superpositioned by trivial supervertices consisting of a single vertex with three incident semiedges, and such edges by trivial superedges consisting of a single isolated edge.

The following theorem, as established in [11], applies to snarks of girth ≥ 5 that are cyclically 4-edge connected. However, it is worth noting that the result remains valid for snarks with smaller girths as well.

Theorem 1.6. *For a snark G, any proper superposition $G(\mathcal{V}, \mathcal{E})$ is also a snark.*

Normal colorings of superpositioned snarks. Normal 5-edge-colorings for certain families of superpositioned snarks are analyzed in the series of papers [24–26]. All three papers consider snarks obtained from a snark G by superpositioning the vertices and edges of a cycle C of G by supervertices A or A' and by superedges of the form $H_{x,y}$ where H is any snark and x, y a pair of nonadjacent vertices of H. In all three papers, the same approach is used, where it is assumed that G does have a normal 5-edge-coloring σ and this coloring is then extended to a superposition. Papers [24] and [26] investigate the case when $H = P_{10}$ for every edge of a cycle C. With such a superposition, the problem with the construction of a normal 5-edge-colorings arises when C is an odd-length cycle.
It is established that the problem is not inherent to the approach, as the example of a snark G and its superposition is provided, where the superposition does not have a normal 5-edge-coloring, which is an extension of the coloring of G. Instead, the problem arises due to $H = P_{10}$ being a small snark of the diameter only two. Hence, in [25] the approach is extended to a superopsition by any snark H and any pair of vertices x, y of H with $d(x, y) \geq 3$. Here, two sufficient conditions are given under which a superposition does have a normal 5-edge-coloring are given, the first one is applied to some superpositions by Hypohamiltonian snarks and the other to all superpositions by Flower snarks, showing thus that all these superpositions have a normal 5-edge-coloring, thus the Petersen Coloring Conjecture is verified for them. Since the Petersen Coloring Conjecture implies the Ford-Fulkerson Conjecture, these results immediately yield the results of [14]. In this paper, we give an overview of all these results.

2 Preliminaries

Let G be a snark, and let $C = u_0 u_1 \cdots u_{g-1} u_0$ represent a cycle of length g in G. The edges of the cycle C are denoted as $e_i = u_i u_{i+1}$ for $i = 0, \ldots, g-1$, where indices are taken modulo g. Additionally, let v_i denote the neighbor of u_i that is distinct from u_{i-1} and u_{i+1}, and define $f_i = u_i v_i$.
The supervertices A and A' are defined as shown in Figure 1. For superedges, we use $H_{x,y}$, where H is a snark and x, y are two non-adjacent vertices of H. Observe that $H_{x,y}$ contains a pair of 3-connectors, denoted S_x and S_y. These connectors consist of the three semiedges that correspond to halves of the three edges in H incident to x and y, respectively. We now formally define the type of superpositions considered in this paper.

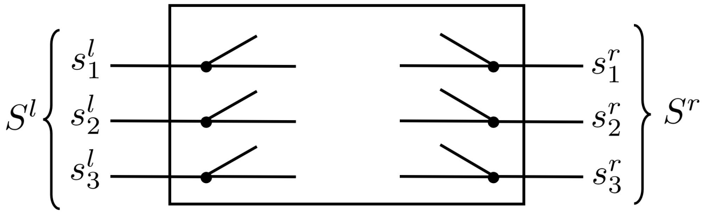

Figure 2: A schematic representation of a superedge $\mathcal{B}_i$ in a superposition $G_C(\mathcal{A}, \mathcal{B})$, showing its left and right connectors along with their semiedges. This representation will be assumed throughout the paper.

Definition 2.1. Let $C = u_0 u_1 \cdots u_{g-1} u_0$ be a cycle in a snark G, and let $e_i = u_i u_{i+1}$ represent an edge of C for $i = 0, \ldots, g-1$. A *superpositioned snark* $G_C(\mathcal{A}, \mathcal{B})$ is a superposition of G such that:

- For every vertex u_i of C, $\mathcal{A}(u_i) \in \{A, A'\}$.
- For every edge e_i of C, $\mathcal{B}(e_i) \in \{H_{x,y} : H$ is a snark and x, y are non-adjacent vertices of $H\}$.

All other vertices and edges of G are superpositioned by themselves.

Note that the snark H used to construct a superedge $\mathcal{B}_i$ does not need to be the same for different edges of the cycle C. The family of all such superpositions is denoted by $\mathcal{G}_C(\mathcal{A}, \mathcal{B})$. For simplicity, we will write $\mathcal{A}_i$ instead of $\mathcal{A}(u_i)$ and $\mathcal{B}_i$ instead of $\mathcal{B}(e_i)$.
In a superedge $\mathcal{B}_i$, the connector that is matched with a connector of $\mathcal{A}_i$ will be referred to as the *left connector* and denoted by S^l, while the connector matched with $\mathcal{A}_{i+1}$ will be called the *right connector* and denoted by S^r. The three semiedges belonging to the left connector are called the *left semiedges*, labeled as s_1^l, s_2^l, and s_3^l. Similarly, the semiedges of the right connector are the *right semiedges*, denoted as s_1^r, s_2^r, and s_3^r. Figure 2 illustrates the schematic structure of a superedge that will be used consistently throughout this paper.
When a superedge $\mathcal{B}_i$ is derived from a snark H by removing two non-adjacent vertices x and y, it is written as $\mathcal{B}_i = H_{x,y}$ if the left connector corresponds to S_x, and as $\mathcal{B}_i = H_{y,x}$ if the left connector corresponds to S_y.
It is useful to assume that the connection between a supervertex $\mathcal{A}_i$ and the superedges $\mathcal{B}_{i-1}$ and $\mathcal{B}_i$ is established as follows: first, the right semiedges of $\mathcal{B}_{i-1}$ are matched with the left semiedges of $\mathcal{B}_i$. Then, one of the resulting edges is subdivided to create the vertex u_i of $\mathcal{A}_i$. Since the identification of

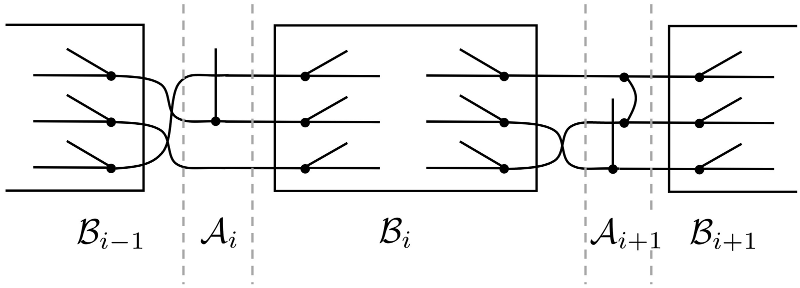

Figure 3: This figure demonstrates how a superedge $\mathcal{B}_i$ is connected to its neighboring superedges $\mathcal{B}_{i-1}$ and $\mathcal{B}_{i+1}$, based on a permutation p_i and a dock d_i associated with $\mathcal{B}_i$. In this example, $p_i = (1, 3, 2)$ and $d_i = 2$. The figure also reveals that $p_{i-1} = (2, 3, 1)$ and $d_{i+1} = 3$.

semiedges between $\mathcal{B}_{i-1}$ and $\mathcal{B}_i$ can occur in multiple ways, as illustrated in Figure 3, we associate a permutation p_{i-1} of the set $\{1, 2, 3\}$ with the superedge $\mathcal{B}_{i-1}$. This permutation specifies how the right semiedges s^r_1, s^r_2, s^r_3 of $\mathcal{B}_{i-1}$ are reordered before being matched with the left semiedges s^l_1, s^l_2, s^l_3 of $\mathcal{B}_i$. Specifically, the semiedge $s^r_{p^{-1}_{i-1}(j)}$ of $\mathcal{B}_{i-1}$ is matched with the semiedge s^l_j of $\mathcal{B}_i$. The permutation p_{i-1} is referred to as a *semiedge permutation.*
For instance, in Figure 3, the semiedge permutations are $p_{i-1} = (2, 3, 1)$ and $p_i = (1, 3, 2)$. When the specific permutation p_{i-1} is clear from the context, we will write $p^{-1}_{i-1}(j)$ simply as j^- for brevity.
Suppose that the edge created by semiedge identification is denoted by $s^r_{j^-}s^l_j$. Among these edges, one $j \in \{1, 2, 3\}$ is selected as the index of the edge to be subdivided to form the vertex u_i. Since j corresponds to a left semiedge in $\mathcal{B}_i$, this choice is represented by $j = d_i$ and is associated with $\mathcal{B}_i$. The value d_i, which determines the left semiedge of $\mathcal{B}_i$ to be connected to the vertex u_i of $\mathcal{A}_i$, is called the *dock* index, and the semiedge $s^l_{d_i}$ is referred to as the *dock* semiedge. For example, in Figure 3, the dock indices are $d_i = 2$ and $d_{i+1} = 3$.
To summarize, each superedge $\mathcal{B}_i$ is associated with a permutation p_i, which determines how the right semiedges of $\mathcal{B}_i$ connect to the left semiedges of $\mathcal{B}_{i+1}$, and a dock index d_i, which specifies which left semiedge of $\mathcal{B}_i$ connects to the vertex u_i of $\mathcal{A}_i$.

Submultipoles and their compatible colorings. We now focus on normal 5-edge-colorings of a superposition $G_C(\mathcal{A}, \mathcal{B})$. To proceed, we first introduce the concepts of a submultipole and the restriction of a coloring to

a submultipole.
Let $M = (V, E, S)$ be a multipole. A multipole $M' = (V', E', S')$ is called a *submultipole* of M if $V' \subseteq V$, $E' \subseteq E$, and $S' \subseteq S \cup E_S$, where E_S represents the set of halves of edges in E. For a subset $V' \subseteq V$, a multipole $M' = M[V']$ is called an *induced* submultipole of M if:

- The vertex set is V'.
- The edge set E' includes all edges $e \in E$ where both endpoints belong to V'.
- The semiedge set S' includes all semiedges in M with endpoints in V', along with the halves of edges in E that have exactly one endpoint in V'.

Now, let σ be a normal 5-edge-coloring of a cubic multipole M. The *restriction* of σ to a submultipole M', denoted $\sigma' = \sigma|_{M'}$, is defined as follows:

- For each edge $e \in E'$, $\sigma'(e) = \sigma(e)$.
- For each semiedge $s \in S' \cap S$, $\sigma'(s) = \sigma(s)$.
- For each semiedge $s \in S' \setminus S$, $\sigma'(s) = \sigma(e_s)$, where e_s is the edge in M whose semiedge is s.

Finally, let $M_1, \dots, M_k$ be cubic submultipoles of a cubic multipole M, and let σ_i be a normal 5-edge-coloring of M_i for $i = 1, \dots, k$. Let M' denote the submultipole of M induced by the union of vertices $\cup_{i=1}^{k} V(M_i)$. The colorings σ_i are said to be *compatible* if there exists a normal 5-edge-coloring σ' of M' such that $\sigma'|_{M_i} = \sigma_i$ for every $i = 1, \dots, k$.

Our approach to the coloring of a superposition Throughout the paper, we assume that a snark G has a normal 5-edge-coloring σ, and we wish to extend this coloring to a superposition $\tilde{G} \in \mathcal{G}_C(\mathcal{A}, \mathcal{B})$. To be more precise, let C be a cycle in G and denote by M_{int} a submultipole of G induced by $V(G) \backslash V(C)$. By $\tilde{\sigma}_{\text{int}}$ we denote the restriction of σ to the submultipole M_{int}. Notice that M_{int} is a submultipole of the superposition $\tilde{G}$ also. We aim to construct a normal 5-edge coloring $\tilde{\sigma}$ of a superposition $\tilde{G}$ such that the restriction of $\tilde{\sigma}$ to M_{int} equals $\tilde{\sigma}_{\text{int}}$. We achieve this by constructing a normal 5-edge-coloring $\tilde{\sigma}_i$ of each superedge $\mathcal{B}_i$ with particular properties which assure that $\tilde{\sigma}_{i-1}$, $\tilde{\sigma}_i$ and $\tilde{\sigma}_{\text{int}}$ are compatible in $\tilde{G}$ for every i. This will directly imply the compatibility of all $\tilde{\sigma}_i$ and $\tilde{\sigma}_{\text{int}}$, i.e. the existance of a normal 5-edge-coloring of $\tilde{G}$.

3 Superposition by the Petersen graph

The Petersen graph, being the smallest snark, serves as a natural starting point for our study of superpositions where superedges are derived from this graph. Specifically, in this section, we consider the superpositions defined in Definition 2.1, where $H = P_{10}$ for each edge $e_i \in E(C)$, i.e., $\mathcal{B}_i = (P_{10})_{u,v}$ for all $i = 0, \dots, g-1$.

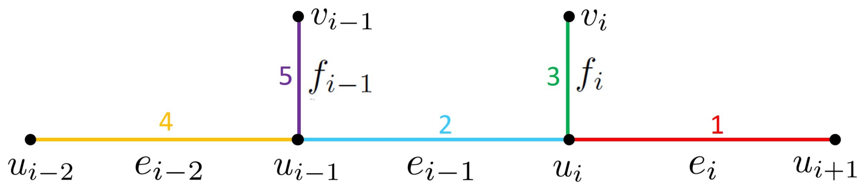

Figure 4: A normal 5-edge-coloring σ of the edges in G incident to the vertices u_{i-1} and u_i.

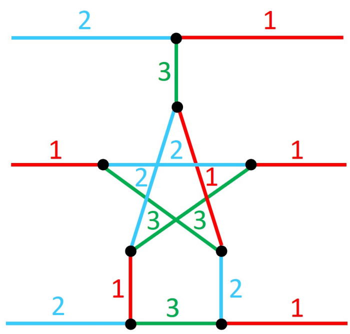

Figure 5: A normal 5-edge-coloring of $\mathcal{B}_i$ that is both right-side σ-monochromatic and left-side σ-compatible, assuming $d_i = 2$.

To proceed, we first define the concept of the color scheme of a semiedge. For a semiedge s in a cubic multipole M, the *color scheme* $\sigma[s]$ is given by $\sigma[s] = (i, \{j, k\})$, where i is the color of s and $\{j, k\}$ is the set of the two colors on the (semi)edges adjacent to s. Now, let s' be another semiedge in a multipole with the color scheme $\sigma[s'] = (i', \{j', k'\})$. The color schemes $\sigma[s]$ and $\sigma[s']$ are said to be *consistent*, denoted by $\sigma[s] \approx \sigma[s']$, if $i' = i$ and either $\{j', k'\} = \{j, k\}$ or $\{j', k'\} \cap \{i, j, k\} = \emptyset$. If the color schemes $\sigma[s]$ and $\sigma[s']$ are consistent, then identifying the semiedges s and s' when "gluing" two multipoles results in either a poor or rich edge.

Next, we define a specific type of coloring for superedges used in our construction. A normal 5-edge-coloring $\tilde{\sigma}_i$ of $\mathcal{B}_i$ is called *right-side σ-monochromatic* if for every $j \in \{1,2,3\}$, the condition $\tilde{\sigma}_i[s^r_j] \approx (\sigma(e_i), \{\sigma(e_{i-1}), \sigma(f_i)\})$ holds. For instance, if the edge-coloring σ of G around vertex u_i is as shown in Figure 4, then the coloring of $\mathcal{B}_i$ depicted in Figure 5 is an example of a right-side σ-monochromatic coloring of $\mathcal{B}_i$.
A normal 5-edge-coloring $\tilde{\sigma}_i$ of $\mathcal{B}_i$ is said to be *left-side σ-compatible* if the following conditions are satisfied:

- $\tilde{\sigma}_i[s^l_{d_i}] \approx (\sigma(e_i), \{\sigma(e_{i-1}), \sigma(f_i)\})$,
- $\tilde{\sigma}_i[s^l_j] \approx (\sigma(e_{i-1}), \{\sigma(e_i), \sigma(f_i)\})$ for every $j \in \{1,2,3\} \setminus \{d_i\}$, and
- there exists a Kempe $(\sigma(e_{i-1}), \sigma(f_i))$-chain P^l that connects the two left semiedges s^l_j where $j \neq d_i$.

To illustrate this concept, assume the edge-coloring σ of G is as shown in Figure 4. If $d_i = 2$, then the coloring of $\mathcal{B}_i$ depicted in Figure 5 is an example of a left-side σ-compatible coloring of $\mathcal{B}_i$.

Remark 3.1. Let $\tilde{\sigma}_{i-1}$ be a right-side σ-monochromatic coloring of $\mathcal{B}_{i-1}$, $\tilde{\sigma}_i$ a left-side σ-compatible coloring of $\mathcal{B}_i$, and $\tilde{\sigma}_{\text{int}}$ the restriction of σ to M_{int}. The following holds:

- If $\mathcal{A} = A$, then $\tilde{\sigma}_{i-1}$, $\tilde{\sigma}_i$, and $\tilde{\sigma}_{\text{int}}$ are compatible, meaning they combine seamlessly.
- If $\mathcal{A} = A'$, compatibility still holds, but $\tilde{\sigma}_i$ must be replaced by a modified coloring $\tilde{\sigma}'_i$, which is obtained by swapping colors along the Kempe chain P^l.

Therefore, if every superedge $\mathcal{B}_i$ admits a normal 5-edge-coloring that is simultaneously right-side σ-monochromatic and left-side σ-compatible, a normal 5-edge-coloring of the entire superposition can be achieved. However, when $d_i = 1$, such a coloring of $\mathcal{B}_i$ would result in the semiedges s^l_1 and s^r_1 having the same color. Since these semiedges are incident to the same vertex, this would violate the condition of a proper coloring. Thus, we conclude the following.

Observation 3.2. *A superedge $\mathcal{B}_i$ cannot admit a normal 5-edge-coloring that is simultaneously right-side σ-monochromatic and left-side σ-compatible in the case where $d_i = 1$.*

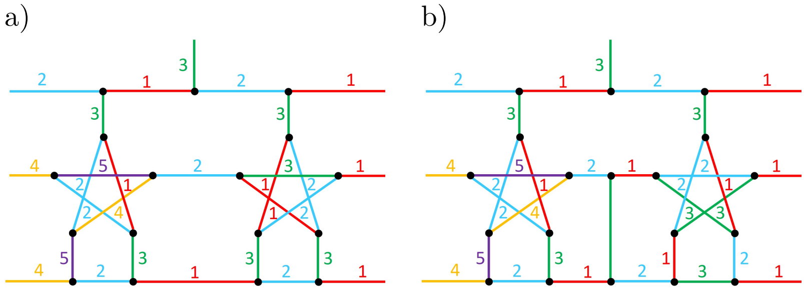

Figure 6: Given a coloring σ of G as shown in Figure 4, this figure illustrates σ-compatible colorings: $\tilde{\sigma}_{i-1}$, which is left-side σ-compatible, and $\tilde{\sigma}_i$, which is right-side σ-monochromatic. The two cases shown are: a) $\mathcal{A}_i = A$ and b) $\mathcal{A}_i = A'$, both with $d_{i-1} = d_i = 1$.

To address this issue, we group certain pairs of consecutive superedges into larger "chunks." By doing so, it becomes possible to ensure that these larger chunks are both right-side σ-monochromatic and left-side σ-compatible. This concept is illustrated in Figures 6 and 7. Using this strategy, we can demonstrate that when $d_i \neq 1$ for at least one superedge $\mathcal{B}_i$, the superedges of the superposition $G_C(\mathcal{A}, \mathcal{B})$ can be partitioned into a combination of single superedges and pairs of consecutive superedges, such that:

- A single superedge is assigned a normal 5-edge-coloring that is both right-side σ-monochromatic and left-side σ-compatible, as shown in Figure 5.
- A pair of consecutive superedges forms a larger chunk that is colored to be right-side σ-monochromatic and left-side σ-compatible, as depicted in Figures 6 and 7.

With additional refinements, this approach leads to the following result:

Theorem 3.3. *[26] Let G be a snark, σ a normal 5-edge-coloring of G, C a cycle of length g in G, and $\tilde{G} \in \mathcal{G}_C(\mathcal{A}, \mathcal{B})$ a superposition of G. If there exists at least one $i \in \{0, \ldots, g-1\}$ such that $p_i(1) \neq 1$ or $d_i \neq 1$, then $\tilde{G}$ admits a normal 5-edge-coloring $\tilde{\sigma}$ with at least 18 poor edges.*

On the other hand, if $d_i = 1$ for every superedge $\mathcal{B}_i$, the above approach succeeds only when C is an even-length cycle. This limitation arises because an odd number of superedges cannot be grouped into pairs for coloring as shown in Figure 6, and a single superedge $\mathcal{B}_i$ with $d_i = 1$ cannot have

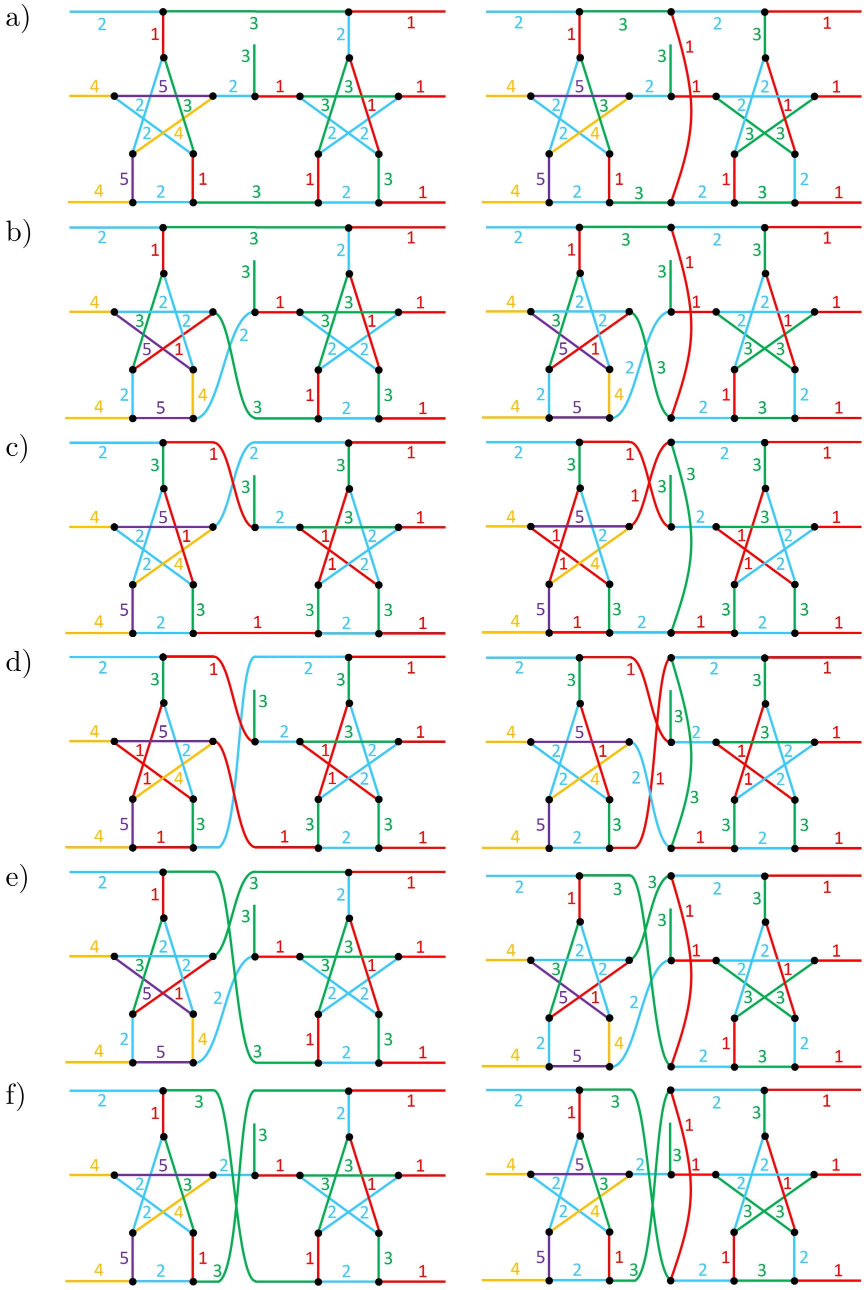

Figure 7: For the normal 5-edge-coloring σ of G shown in Figure 4, this figure illustrates normal 5-edge-colorings of $\mathcal{B}_{i-1}$ and $\mathcal{B}_i$ that are compatible with σ. The left column corresponds to $\mathcal{A}_i = A$, and the right column corresponds to $\mathcal{A}_i = A'$. These configurations are shown for $d_i = 2$ and the following permutations p_{i-1}: a) $(1,2,3)$, b) $(1,3,2)$, c) $(2,1,3)$, d) $(2,3,1)$, e) $(3,1,2)$, and f) $(3,2,1)$.

a normal 5-edge-coloring that is both right-side σ-monochromatic and left-side σ-compatible, as stated in Observation 3.2. Consequently, in such cases, the following theorem provides the best result achievable by our approach:

Theorem 3.4. *[24] Let G be a snark, σ a normal 5-edge-coloring of G, C a cycle of length g in G, and $\tilde{G} \in \mathcal{G}_C(\mathcal{A}, \mathcal{B})$ a superposition of G. If $p_i = (1, 2, 3)$ and $d_i = 1$ for every $i \in \{0, \ldots, g-1\}$, then for even g, there exists a normal 5-edge-coloring $\tilde{\sigma}$ of $\tilde{G}$ with at least 18 poor edges.*

Next, we consider the case of a superposition along an odd-length cycle C in a snark G, where $p_i = (1, 2, 3)$ and $d_i = 1$ for every superedge $\mathcal{B}_i$. The question arises whether it is generally impossible to extend a normal 5-edge-coloring of G to such a superposition $\tilde{G}$, or if this limitation is specific to our approach. To explore this, we analyze the Petersen graph $G = P_{10}$.
The Petersen graph G, being vertex-transitive, has (up to isomorphism) a single normal 5-edge-coloring, in which every edge is rich. It can be verified computationally that the following holds:

Observation 3.5. *[24] Let G be the Petersen graph and C a cycle of length 5 in G. Consider the superposition $G_C(\mathcal{A}, \mathcal{B})$ of G such that $\mathcal{A}_i = A$ and $\mathcal{B}_i = (P_{10})_{u,v}$ with $p_i = (1, 2, 3)$ and $d_i = 1$ for all $i \in \{0, \ldots, 4\}$. It is not possible to extend the normal 5-edge-coloring of G to the superposition $G_C(\mathcal{A}, \mathcal{B})$ without altering the colors of edges in G outside C.*

It is important to note that the above observation does not contradict the Petersen Coloring Conjecture. The superposition $G_C(\mathcal{A}, \mathcal{B})$ does admit a normal 5-edge-coloring; however, this coloring cannot be obtained as an extension of the normal 5-edge-coloring of $G = P_{10}$. In other words, while the superposition can be colored normally, doing so requires changing the colors of edges in G outside the cycle C.

4 Superposition by any snark

In the approach discussed in the previous section, a problem arises when the Petersen graph P_{10} is used as a superedge due to its diameter being two. This implies that any two vertices in P_{10} are at a distance of at most 2. Consequently, for at least one choice of the dock d_i, an edge of the snark G belonging to the cycle C is replaced in the superposition by a path of length two. It is evident that such a path cannot be assigned the same color as the corresponding edge in C without violating the properness of the coloring.
To address this issue, we use larger snarks as superedges and restrict attention to pairs of vertices u, v in these snarks such that $d(u, v) \geq 3$. In other

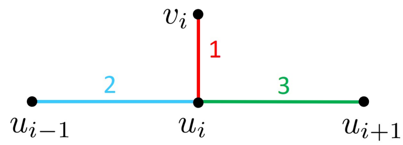

Figure 8: A coloring σ of the edges incident to a vertex u_i of the cycle C in G. For this coloring σ, all colorings of $\mathcal{B}_i$ consistent with the color schemes shown in Figure 9 are σ-compatible.

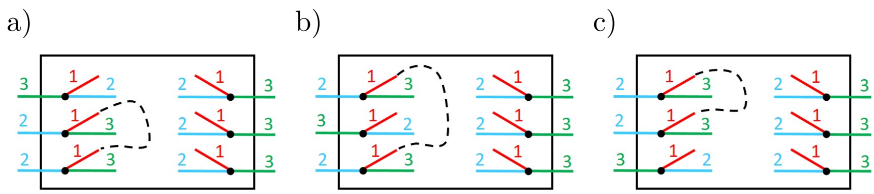

Figure 9: The color scheme κ_j for: a) $j = 1$, b) $j = 2$, c) $j = 3$. The dashed curve represents the Kempe chain P^l.

words, we focus on superedges of the form $H_{u,v}$, where H is a snark with a diameter of at least three, and u, v are vertices in H satisfying $d(u, v) \geq 3$. To generalize the approach from the previous section for any snark used as a superedge, we extend the notion of (consistent) color schemes from semiedges to connectors and superedges. Let $\tilde{G} \in \mathcal{G}_C(\mathcal{A}, \mathcal{B})$ be a superposition of G, $\mathcal{B}_i$ a superedge of $\tilde{G}$, and $\tilde{\sigma}_i$ a 5-edge-coloring of $\mathcal{B}_i$. The *color scheme* of the left and right connector of $\mathcal{B}_i$ is defined as follows:

$$\tilde{\sigma}_i[S^l] = (\tilde{\sigma}_i[s_1^l], \tilde{\sigma}_i[s_2^l], \tilde{\sigma}_i[s_3^l]) \quad \text{and} \quad \tilde{\sigma}_i[S^r] = (\tilde{\sigma}_i[s_1^r], \tilde{\sigma}_i[s_2^r], \tilde{\sigma}_i[s_3^r]),$$

respectively. The *color scheme* of a superedge $\mathcal{B}_i$ is then defined as:

$$\tilde{\sigma}_i[\mathcal{B}_i] = (\tilde{\sigma}_i[S^l], \tilde{\sigma}_i[S^r]),$$

and is illustrated in Figure 8.
Let $\mathcal{B}_i$ be a superedge of $G_C(\mathcal{A}, \mathcal{B})$, and let $\tilde{\sigma}_i$ and $\tilde{\sigma}'_i$ be two normal 5-edge-colorings of $\mathcal{B}_i$. The colorings $\tilde{\sigma}_i$ and $\tilde{\sigma}'_i$ are *consistent* on the left connector S^l of $\mathcal{B}_i$, denoted by $\tilde{\sigma}_i[S^l] \approx \tilde{\sigma}'_i[S^l]$, if:

$$\tilde{\sigma}_i[s_j^l] \approx \tilde{\sigma}'_i[s_j^l] \quad \text{for all } j = 1, 2, 3.$$

The consistency of $\tilde{\sigma}_i$ and $\tilde{\sigma}'_i$ on the right connector S^r of $\mathcal{B}_i$ is defined analogously. Finally, the colorings $\tilde{\sigma}_i$ and $\tilde{\sigma}'_i$ are said to be *consistent* on $\mathcal{B}_i$, denoted by $\tilde{\sigma}_i[\mathcal{B}_i] \approx \tilde{\sigma}'_i[\mathcal{B}_i]$, if they are consistent on both the left and right connectors. When this condition holds, we also say that the color schemes $\tilde{\sigma}_i[\mathcal{B}_i]$ and $\tilde{\sigma}'_i[\mathcal{B}_i]$ are consistent.

4.1 Right colorings

A normal 5-edge-coloring of $\mathcal{B}_i$ is called *j-right* if it is consistent with the color scheme κ_j from Figure 9 and there exists a Kempe $(2,1)$-chain P^l connecting the pair of left semiedges distinct from s^l_j. A superedge $\mathcal{B}_i$ is classified as follows:

- *Dock-right*: If it is j-right for $j = d_i$.
- *Doubly-right*: If it is j-right for at least two distinct values of j.
- *Fully-right*: If it is j-right for all $j \in \{1, 2, 3\}$.

A j-right coloring of $\mathcal{B}_i$ is σ-compatible with the coloring σ of the cycle C in G, as shown in Figure 8, provided that the dock of $\mathcal{B}_i$ is $d_i = j$. For any other coloring σ of G, a σ-compatible coloring of $\mathcal{B}_i$ can be derived from a j-right coloring by applying a color permutation and/or swapping colors along P^l. Additionally, a j-right coloring of $\mathcal{B}_i$ is compatible with j-right colorings of $\mathcal{B}_{i-1}$ and $\mathcal{B}_{i+1}$, provided all these colorings are σ-compatible. Based on this, we establish the following theorem:

Theorem 4.1. *[25] Let G be a snark with a normal 5-edge-coloring σ, C a cycle of length g in G, and $\tilde{G} \in \mathcal{G}_C(\mathcal{A}, \mathcal{B})$ a superposition of G. If every superedge $\mathcal{B}_i$ is dock-right for $i = 0, \ldots, g-1$, then $\tilde{G}$ admits a normal 5-edge-coloring.*

As an immediate consequence of Theorem 4.1, we obtain the following corollary:

Corollary 4.2. *[25] Let G be a snark with a normal 5-edge-coloring σ, C a cycle of length g in G, and $\tilde{G} \in \mathcal{G}_C(\mathcal{A}, \mathcal{B})$ a superposition of G. If every superedge $\mathcal{B}_i$ is fully-right for $i = 0, \ldots, g-1$, then $\tilde{G}$ admits a normal 5-edge-coloring.*

The sufficient conditions outlined in Theorem 4.1 and Corollary 4.2 hold for any snark H used as a superedge and for any pair of vertices u, v in H such that $d(u, v) \geq 3$. To demonstrate the broad applicability of these conditions, in the next subsection we describe a large family of snarks H for which these conditions are satisfied.
Before proceeding, we note that the extension of a coloring σ of a snark G to a superposition $\tilde{G}$ can be performed independently along multiple vertex-disjoint cycles. Recall that an *even subgraph* of a graph G is a subgraph

where every vertex has an even degree. Based on this, we state the following formal remark:

Remark 4.3. Theorem 4.1 remains valid when C is an even subgraph of G.

4.2 Right colorings and hypohamiltonian snarks

We now demonstrate that the condition in Theorem 4.1 holds for hypohamiltonian snarks. A graph H is said to be *hypohamiltonian* if H itself is not hamiltonian, but the removal of any vertex $v \in V(H)$ results in a graph $H - v$ that is hamiltonian. It is known that an infinite family of snarks, the so-called Flawer snarks, are hypohamiltonian [6]. Furthermore, in [17] hypohamiltonian snarks with cyclic connectivity 5 and 6 are constructed for all but finitely many even orders. Thus, there exist infinitely many snarks for which the following proposition holds.

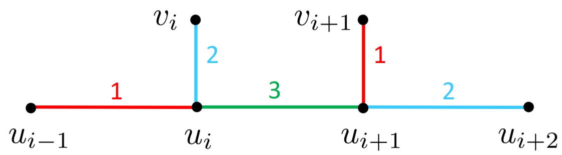

Figure 10: A coloring σ of the edges incident to vertices u_i and u_{i+1} of a cycle C in G. All j,k-left colorings of $\mathcal{B}_i$ consistent with the color schemes in Figure 11 are σ-compatible with this coloring σ of G.

Proposition 4.4. *[25] Let H be a hypohamiltonian snark, and let x, y be a pair of non-adjacent vertices in H. Then, $H_{x,y}$ is j-right for at least one $j \in \{1, 2, 3\}$.*

We outline the proof of this result. For any hamiltonian cycle in $H - y$, the cycle must be of odd length. Thus, its edges can be alternately colored by 1 and 2, except for the two edges incident to x, which are both colored by 2. All other edges in H are assigned the color 3. Removing the vertices x and y from H to obtain $H_{x,y}$, and preserving the colors of (semi)edges in $H_{x,y}$ as in H, results in a j-right coloring of $H_{x,y}$ for some $j \in \{1, 2, 3\}$. Note, however, that the specific value of j is not determined.
As a consequence, Theorem 4.1 and Proposition 4.4 together imply that a superposition admits a normal 5-edge-coloring for any hypohamiltonian snark H used as a superedge, and for any pair of vertices $x, y \in H$ such that $d(x, y) \geq 3$. However, this result applies only to certain ways of identifying

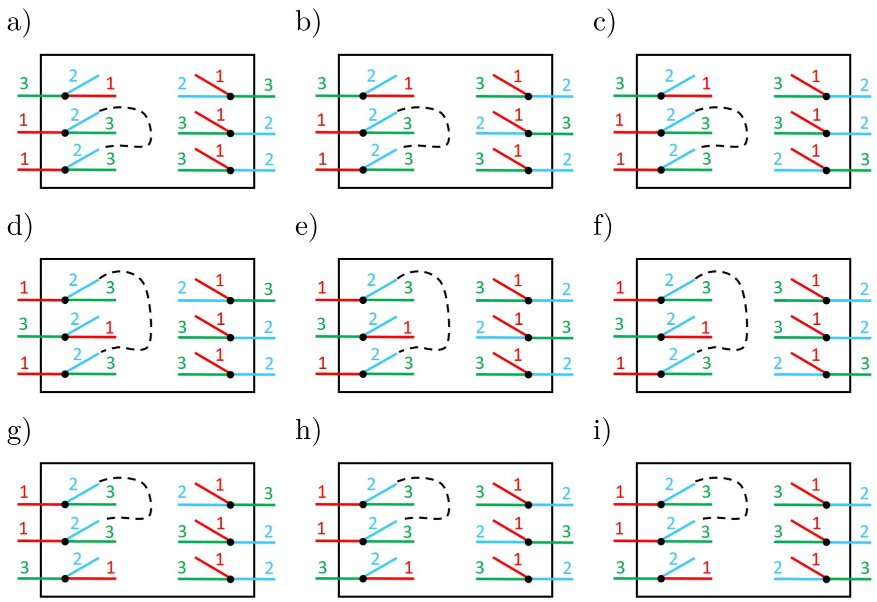

Figure 11: The color scheme $\tau_{j,k}$ for: a) $(j,k) = (1,1)$, b) $(j,k) = (1,2)$, c) $(j,k) = (1,3)$, d) $(j,k) = (2,1)$, e) $(j,k) = (2,2)$, f) $(j,k) = (2,3)$, g) $(j,k) = (3,1)$, h) $(j,k) = (3,2)$, i) $(j,k) = (3,3)$.

semiedges. Moreover, it can be verified that for some choices of x, y already in H being the smallest Flower snark, the superedge $H_{x,y}$ is not fully-right. Hence, a more refined sufficient condition is needed.

4.3 Left colorings

A normal 5-edge-coloring of $\mathcal{B}_i$ is referred to as a *j,k-left coloring* if it is consistent with the color scheme $\tau_{j,k}$ shown in Figure 11 and there exists a Kempe $(1,2)$-chain P^l connecting a pair of left semiedges distinct from s_j^l. A superedge $\mathcal{B}_i$ is called *doubly-left* if, for every $j \in \{1,2,3\}$, it admits a j,k-left coloring for at least two distinct values of k. In other words, $\mathcal{B}_i$ is doubly-left if it has a normal 5-edge-coloring consistent with at least two color schemes from each row in Figure 11.
For a coloring σ of G as illustrated in Figure 10, a j,k-left coloring of $\mathcal{B}_i$ is σ-compatible, provided that $d_i = j$. If $\mathcal{A} = A$, such a coloring is also compatible with a right coloring of $\mathcal{B}_{i-1}$. If $\mathcal{A} = A'$, compatibility with a right coloring of $\mathcal{B}_{i-1}$ can be achieved by swapping colors along P^l. For other colorings σ of G, a σ-compatible coloring of $\mathcal{B}_i$ with similar properties can be obtained from a j,k-left coloring by applying a color permutation

and/or appropriate color swaps along P^l.
By leveraging the compatibility of left and right colorings, and the compatibility among right colorings, and partitioning the superedges into singletons or consecutive pairs, we establish the following result:

Theorem 4.5. *[25] Let G be a snark with a normal 5-edge-coloring σ, C a cycle of length g in G, and $\tilde{G} \in \mathcal{G}_C(\mathcal{A}, \mathcal{B})$ a superposition of G. If each superedge $\mathcal{B}_i$ is both doubly-right and doubly-left for every $i = 0, \ldots, g-1$, then $\tilde{G}$ admits a normal 5-edge-coloring.*

As with Theorem 4.1, Theorem 4.5 provides a sufficient condition for extending a normal 5-edge-coloring of G to its superposition. This condition applies to superedges of the form $H_{x,y}$, where H is any snark, and x, y are any vertices in H satisfying $d(x, y) \geq 3$. To illustrate the broad applicability of this condition, we present an infinite family of snarks to which it applies, namely, Flower snarks.
Furthermore, since the extension described in Theorem 4.5 can be applied independently along multiple vertex-disjoint cycles, it follows that Theorem 4.5 also holds when C is an even subgraph of G.

4.4 Left colorings and Flower snarks

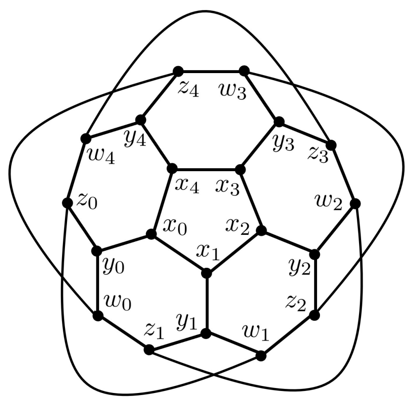

Figure 12: The Flower snark J_5.

A Flower snark J_r, for odd $r \geq 5$, is defined as a graph with the vertex set

$$V(J_r) = \bigcup_{i=0}^{r-1} \{x_i, y_i, z_i, w_i\},$$

and the edge set

$$E(J_r) = \bigcup_{i=0}^{r-1} \{x_i x_{i+1}, x_i y_i, y_i z_i, y_i w_i, z_i w_{i+1}, w_i z_{i+1}\},$$

where indices are taken modulo r. The Flower snark for $r = 5$ is depicted in Figure 12. We consider a superedge $H_{x,y}$, where $H = J_r$, and $x, y \in V(H)$ is any pair of vertices in H such that $d(x, y) \geq 3$.
To apply Theorem 4.5 to all superpositions using Flower snarks, we need to establish that the superedge $(J_r)_{x,y}$ is both doubly-right and doubly-left for every odd $r \geq 5$ and every pair of vertices x, y in J_r with $d(x, y) \geq 3$. A reduction method introduced by Hagglund and Steffen [7] allows us to limit our consideration to the Flower snark J_5 and specific pairs of vertices in J_7, as detailed in the following proposition, which is verified computationally (*in silico*).

Proposition 4.6. *[25] Let J_5 be the Flower snark, and let x, y be a pair of vertices in J_5 such that $d(x, y) \geq 3$. Then the superedge $(J_5)_{x,y}$ is both doubly-right and doubly-left. The same holds for a superedge $(J_7)_{x,y}$, where $(x, y) \in \{(x_0, x_3), (z_0, z_3), (z_3, z_0)\}$.*

Building on the above proposition and using the reduction method for larger Flower snarks, the application of Theorem 4.5 leads to the following result:

Theorem 4.7. *[25] Let G be a snark with a normal 5-edge-coloring σ, C an even subgraph of G, and $\tilde{G} \in \mathcal{G}_C(\mathcal{A}, \mathcal{B})$ a superposition of G. If $\mathcal{B}_i \in \{(J_r)_{x,y} : x, y \in V(J_r) \text{ and } d(x, y) \geq 3\}$ for every $e_i \in E(C)$, then $\tilde{G}$ admits a normal 5-edge-coloring.*

Since the Petersen Coloring Conjecture implies the Ford-Fulkerson Conjecture, Theorem 4.7 implies the results presented in [14].

5 Concluding remarks

The findings derived from our research on this topic have inspired us to propose a conjecture suggesting the validity of additional claims. To articulate this conjecture, we first introduce some essential definitions. A normal coloring of a cubic graph G is called a *strong coloring* if every edge in the graph is rich. The minimum number of colors needed to achieve a strong coloring of a cubic graph G is referred to as the *strong chromatic index* and is denoted by $\chi'_s(G)$. Let $\mathrm{NC}(G)$ represent the set of all normal 5-colorings of G. The Petersen Coloring Conjecture asserts that $\mathrm{NC}(G) \neq \emptyset$ for any bridgeless cubic graph G.

Assuming the Petersen Coloring Conjecture is true, meaning that $\mathrm{NC}(G)$ is non-empty for every bridgeless cubic graph G, we define $\mathrm{poor}(G)$ as the highest number of poor edges found across all colorings in $\mathrm{NC}(G)$. In examining $\mathrm{poor}(G)$, it is particularly insightful to first analyze the 3-cycles and 4-cycles present in a snark G, provided such cycles exist within G.

Remark 5.1. [24] Let G be a bridgeless cubic graph and σ a normal 5-coloring of G. If G contains a 3-cycle C, then every edge in C is poor in σ. Similarly, if G contains a 4-cycle C, then either 2 or 4 edges of C must be poor in σ.

It follows that any graph admitting a normal 5-coloring devoid of poor edges must have a girth of at least 5. Denote by P_{10}^{Δ} the graph obtained from P_{10} by truncating one of its vertices. A normal 5-edge-coloring of P_{10}^{Δ} has at least 3 poor edges, and a straightforward verification confirms that it has exactly 3 poor edges. This observation motivates the following conjecture.

Conjecture 5.2. [24] Let G be a bridgeless cubic graph. If $G \neq P_{10}$, then $\mathrm{poor}(G) > 0$. Moreover, if $G \neq P_{10}, P_{10}^{\Delta}$, then $\mathrm{poor}(G) \geq 6$.

Regarding the two sufficient conditions for superpositions by any snark H and any pair of vertices x, y in H with $d(x, y) \geq 3$, the sufficient condition in Theorem 4.1 is weaker than that in Theorem 4.5. Nonetheless, it is applicable to superpositions by infinitely many distinct snarks, specifically to all hypohamiltonian snarks used as superedges, although not to all possible ways of semiedge identification. For instance, since Flower snarks are hypohamiltonian, Theorem 4.1 implies that many snarks superposed by Flower snarks admit a normal 5-edge-coloring. In contrast, the condition of Theorem 4.5 is more stringent. When applied to snarks superposed by Flower snarks, it guarantees that all such superpositions have a normal 5-edge-coloring.
For Flower snark superedges, the sufficient condition in Theorem 4.5 can be reformulated to involve the corresponding 2-factors, whose existence is then verified computationally. A promising avenue for future research is to establish that all snarks, or certain broad families of snarks, possess the required 2-factorizations.

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