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Two reminders on Ptolemy and Ramanujan and some problems

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Abstract

We present, discuss, and offer alternative proofs for a couple of beautiful results spanning almost two millennia, but unified by their connections to Indian mathematics. Several open problems are suggested for future research.

1 Introduction

The motivation for this note was the nice Croatian - Indian mathematical evening held on Dec. 20, 2024, at the Department of Mathematics, University of Zagreb, where three 20-minute lectures were given by academician Andrej Dujella, prof. dr. sc. Zvonimir Šikić and prof. dr. sc. Mirko Primc, relating their research works connected to some Indian mathematicians. The main organizer of the event was prof. dr. sc. Darko Žubrinić. After the lectures and some Indian food snacks and Croatian wines and beverages, I put on the blackboard some of Ramanujan's problems. The lecturers and other participants didn't know at the moment how to prove them (neither did I). The first topic of this note (on Ptolemy) is also deeply interconnected with Indian mathematics. So, these are the main motivations of this note. In my translation [16] of the beautiful book [11], in two topics I added some additional new stuff that does not appear in the original (as well as some others). The first is Ptolemy's formula in the topic *Ptolemy's Almagest*, year about 150 and Fuhrmann's formula, and in the year 1500 topic *The series for computing* π, I added a wonderful identity of Ramanujan which I'll explain in the sequel. This note is not only my own research, but I think it's worth reminding us of these two gems of mathematics. We end the paper with some problems (not in the original) from [16].

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2 Ptolemy's theorem

The famous mathematician and astronomer Ptolemy, or Claudius Ptolomaeus (c. 90 - c. 168) from Alexandria, published about the year 150 his comprehensive work *Almagestus*, or simply *Almagest* in 13 books, where he described almost all knowledge of astronomy and mathematics known to his time. The work is also known in Latin as *Syntaxis Mathematica.* He created a geocentric model of the Universe that was accepted as true for more than 1300 years until Copernicus' *Revolution of the Celestial Spheres* in 1543. Ptolemy had trigonometric tables of certain quantities like the function sine with measures of every 15'.
From the tables, he deduced the formula for the sine of the sum of two angles. In fact, this was the root of the theorem, many centuries later named after Ptolemy.
A (convex) quadrilateral (or any convex polygon) is called *cyclic* if it is inscribed in a circle (i.e. all of its vertices lie on a single circle). Now we can formulate the basic theorem.

Theorem 2.1 (Ptolemy's theorem (about AD 150))**.** *A quadrilateral $ABCD$ (vertices in this order) is cyclic if and only if the product of the lengths of its diagonals is equal to the sum of the products of the lengths of the pairs of opposite sides.*

In symbols, if $|XY|$ is the length of the segment between points X and Y, and if we denote $|AB| = a$, $|BC| = b$, $|CD| = c$, $|DA| = d$, $|AC| = e$, $|BD| = f$, then we have:

$$ef = ac + bd. \tag{2.1}$$

It seems that the first rigorous proof of this theorem was given by the Arab mathematician (and translator) Abul Wafa (or Wefa) about AD 980. However, many used the Ptolemy formula much earlier. For example, the Indian mathematician Brahmagupta (598-660) used Ptolemy's theorem to compute the area and the radius of the cyclic quadrilateral in terms of side lengths around year 650. In fact, Brahmagupta first proved (with the same notation as above) that

$$\frac{f}{e} = \frac{ad + bc}{ab + cd}. \tag{2.2}$$

Equations (2.1) and (2.2) enable us to express the lengths of diagonals in terms of side lengths of cyclic quadrilaterals. Then, using the well-known Heron's formula from about AD 60, which gives the triangle area in terms of its side lengths, Brahmagupta computed the area S and the radius R of the cyclic quadrilateral in terms of its side lengths a, b, c, d as

$$16S^2 = (a+b+c-d)(a+b-c+d)(a-b+c+d)(-a+b+c+d), \tag{2.3}$$

and

$$R^2(a+b+c-d)(a+b-c+d)(a-b+c+d)(-a+b+c+d) = \\ = (ab+cd)(ac+bd)(ad+bc). \tag{2.4}$$

Note that Brahmaguptas's formulae reduce to triangle formulae for say, $d = 0$. More precisely, formula (2.3) reduces to Heron's formula and formula (2.4) to the triangle's circumradius. Apparently, some of these formulae for triangles were known even to Archimedes about 220 B.C.

Proof of Ptolemy's theorem. There are many known proofs of Theorem 2.1, as presented e.g. in [1]. Perhaps the shortest and most elegant proof is by inversion. Choose a big circle K with the center, say, D and radius r, so that the circumcircle $k = ABCD$ is inside K. Consider the inversion $I = I(D,r)$. Then k is mapped into a line k'. Let $A' = I(A)$, $B' = I(B)$, $C' = I(C)$. Then $|A'B'| + |B'C'| = |A'C'|$ on the line k'. But

$$|A'B'| = \frac{|AB|r^2}{|DA||DB|},$$

and similarly for $|B'C'|$ and $|A'C'|$. So, equality (2.1) follows.
Conversely, if one of the vertices does not lie on the circle k, say, B, then $|A'B'| + |B'C'| > |A'C'|$, by triangle inequality, hence $ac + bd > ef$.
The formula (2.2) can also be easily proved via inversion. For a more general fact see *Mathologer, Ptolemy's theorem.* □

Ptolemy's theorem is equivalent to the following facts: the addition formulas for sine and cosine, Pythagoras' theorem, the sine law for triangles, the cosine law for triangles, and many more. Since Pythagoras' theorem is equivalent to Euclid's fifth postulate, we may say that Ptolemy's theorem is in the essence of Euclidean geometry.
Today there are many generalizations, extensions, corollaries, and equivalent statements of Ptolemy's theorem beyond those already mentioned. Even some Croatian mathematicians contributed to the topic, e.g. [4, 8, 9]. One of the best-known generalizations and most quoted extensions of Ptolemy's is Fuhrmann's hexagon theorem which I also quoted in [16] in the topic *Ptolemy's Almagest* (150). This theorem (see [2]) is named after the German mathematician Wilhelm Fuhrmann (1833-1904).

Theorem 2.2 (Fuhrmann's theorem (1890))**.** *Let the opposite side lengths of a convex cyclic hexagon be a, a', b, b' and c, c', and let e, f, g be the polygon (big) diagonals, such that a, a' and e have no common polygon vertex, and likewise for b, b' and f and c, c' and g. Then*

$$efg = aa'e + bb'f + cc'g + abc + a'b'c'. \tag{2.5}$$

Idea of the proof. Let $ABCDEF$ be the cyclic hexagon with side lengths $AB = a$, $BC = b$, $CD = c$, $DE = a'$, $EF = b'$, $FA = c'$ and $CF = e$, $DA = f$ and $BE = g$. Here $AB = |AB|$, etc. We apply Ptolemy's theorem to each of the four convex cyclic quadrilaterals $ABDE$, $BCDF$, $ADEF$, and $ABEF$. After some simple algebraic manipulations with Ptolemy's relations, we can obtain the formula (2.5). We omit some tedious computational details. See also some Internet sites such as [2]. □

Ptolemy's theorem in the hyperbolic plane, say with curvature -1, is given by the following. The formula is the same as (2.1), but instead of x now we have $s(x) = \sinh\left(\frac{x}{2}\right)$.

Theorem 2.3 (Ptolemy's theorem in hyperbolic geometry)**.** *Let $ABCD$ be a convex hyperbolic quadrilateral inscribed in a hyperbolic circle. Then*

$$s\left(|AC|\right) s\left(|BD|\right) = s\left(|AB|\right) s\left(|CD|\right) + s\left(|AD|\right) s\left(|BC|\right). \tag{2.6}$$

The converse is also true. A convex hyperbolic quadrilateral $ABCD$ has a hyperbolic circumcircle if three of the points lie on a hyperbolic circle and satisfy equation (2.6).

Proofs are similar to the original proof of Ptolemy's theorem and can be found in the literature cited before. Of course, the spherical version of Ptolemy holds as well with the same formula (on the unit sphere) with $s(x) = \sin\left(\frac{x}{2}\right)$.

In [14], we managed to prove some interesting geometric facts on cyclic pentagons and, among other things, we proved the Robbins formulae which gives a polynomial equation for the area and radius of a cyclic pentagon in terms of its side lengths, something like Brahmagupta's formulas (2.3) for cyclic quadrilateral, but much more involved. A nice survey on the topic of Robbin's conjectures is given in [10]. Recently, D. Svrtan in [13] used Hopf-Wiener factorization of certain Laurent polynomial invariant of cyclic polygons and by tricky computer search obtained huge polynomials for cyclic n-gons areas and circumradius for $n = 4, 5, 6, 7$ and 8.

As said earlier, there are many generalizations of Ptolemy's theorem. The best seems to me is the following from [3].

Theorem 2.4 (M. Bencze, 2011)**.** *Let $A_1A_2, \dots A_n$ be a convex cyclic Euclidean polygon with vertices in given order. Then the following holds*

$$\frac{|A_2A_n|}{|A_1A_2||A_1A_n|} = \frac{|A_2A_3|}{|A_1A_2||A_1A_3|} + \frac{|A_3A_4|}{|A_1A_3||A_1A_4|} + \cdots + \frac{|A_{n-1}A_n|}{|A_1A_{n-1}||A_1A_n|}. \tag{2.7}$$

Proof is again by inversion. Note that for $n = 4$ we get Ptolemy's relation (2.1), for $n = 5$ we have this in [14] in some form, and for $n = 6$ we get Fuhrmann's formula (2.5).
A generalization of Ptolemy's theorem in n-dimensional Euclidean space was given in [5]. Furthermore, a very recent analog of Fuhrmann's theorem in the Lobachevsky plane was given in [7]. Here we stop on Ptolemy.

3 Some Ramanujan identities and conjectures

Now we shall consider a completely different topic, but also deeply connected to Indian mathematics. It is about the brilliant Indian mathematician Srinivasa Ramanujan (1887-1920), see his Collected papers and problems with some solutions [6] having 355 pages (which my colleague M. Primc kindly lent me after our Croatian-Indian math evening).
In [16], I put the following Ramanujan's identity in the article *Series for Computing* π, the year 1500:

$$A + B = \sqrt{\frac{\pi e}{2}}. \tag{3.1}$$

Here A is the infinite series

$$A = 1 + \frac{1}{1 \cdot 3} + \frac{1}{1 \cdot 3 \cdot 5} + \cdots = \sum_{n \geq 1} \frac{1}{(2n-1)!!},$$

and B is the infinite continuous fraction

$$B = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{1 + \cfrac{3}{1 + \cdots}}}}.$$

The exact values of both A and B are not known, but still, their sum is the square root of $\frac{\pi e}{2}$. I have seen it somewhere and couldn't resist but put that gem in my translation [16]. Now I found it in [6], p. 341, as Ramanujan's question 541 in the Indian Journal of Mathematics from 1914. I tried to prove it but with no success. There is no solution in [6]. Then the organizer of that event, my colleague D. Žubrinić, sent me the link of Mathologer `https://www.youtube.com/watch?v=6iTdNmDHfV0` with a very nice explanation and proof of formula (3.1).
So, following this link, I'll try to present proof of 3.1. Once more this proof was shown on *Mathologer Masterclass* on the above link under the title *Ramanujan's easiest hard infinite monster* on June 24, 2023.

Proof of Ramanujan identity (3.1). First, recall the *Gauss normal distribution integral formula.* The area under the Gauss bell is

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-t^2} dt \tag{3.2}$$

See e.g. Wikipedia on *Gaussian integral.* On the other hand, recall *Wallis' formula* from 1665:

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdots}. \tag{3.3}$$

A short proof of Wallis' formula is as follows. It is well known that

$$\sin x = x \prod_{k \geq 1} \left(1 - \left(\frac{x}{k\pi} \right)^2 \right).$$

Substituting x with $\frac{\pi}{2}$ yields formula (3.3). This is a special case of *Euler's product formula*

$$\frac{\sin(\pi z)}{\pi z} = \prod_{n \geq 1} \left(1 - \frac{z^2}{n^2} \right),$$

valid for any complex number z. It is also known as *Euler's sinc function formula* (see e.g., https://proofwiki.org/wiki/Euler_Formula_for_Sine_Function/Complex_Numbers).
Now consider the following series to get A in (3.1). Let

$$y(x) = \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n \geq 1} \frac{x^n}{n!}. \tag{3.4}$$

By taking derivative of (3.4), we have

$$y'(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = 1 + \sum_{n \geq 1} \frac{x^n}{n!}. \tag{3.5}$$

Hence, $y(x) = Ce^x - 1$, but from $y(0) = 0$, we get $C = 1$, so $y(x) = e^x - 1$. So, for $x = 1$, we have

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots ,$$

i.e. the well-known Euler's number e, and the well-known series expansion of the exponential function

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

Now consider the function

$$y(x) = \frac{x}{1} + \frac{x^3}{1\cdot 3} + \frac{x^5}{1\cdot 3\cdot 5} + \frac{x^7}{1\cdot 3\cdot 5\cdot 7} + \cdots = \sum_{n\geq 1} \frac{x^{2n-1}}{(2n-1)!!}. \tag{3.6}$$

By taking the derivative of (3.6), we get the differential equation

$$y'(x) = 1 + xy(x), \tag{3.7}$$

with $y(0) = 0$. The solution of this linear ordinary differential equation of the first order (3.7), as known from the theory of ODE, is given by

$$y(x) = e^{\frac{x^2}{2}} \int_0^x e^{-\frac{t^2}{2}} dt. \tag{3.8}$$

So, the right-hand side of (3.8) is the right-hand side of (3.6). So far we know that

$$\sqrt{\frac{\pi e}{2}} = A + \sqrt{e} \int_1^\infty e^{-\frac{t^2}{2}} dt. \tag{3.9}$$

What is left to prove is that the second summand on the right-hand side of (3.9) is equal to the continuous fraction B from (3.1). Now consider the function

$$y(x) = e^{\frac{x^2}{2}} \sqrt{\frac{\pi}{2}} - e^{\frac{x^2}{2}} \int_0^x e^{-\frac{t^2}{2}} dt. \tag{3.10}$$

By taking the derivative of (3.10), it is easy to check that we get the differential equation (a bit different from (3.7)):

$$y'(x) = xy(x) - 1, \tag{3.11}$$

with $y(0) = \sqrt{\frac{\pi}{2}}$. Keep taking the derivatives of (3.11) repeatedly, we obtain

$$\begin{aligned} y' &= xy - 1 \\ y'' &= xy' + y \\ y''' &= xy'' + 2y \\ y'''' &= xy''' + 3y'' \\ &\vdots \end{aligned}$$

Hence $\frac{y'}{y} = x - \frac{1}{y}$, $\frac{y''}{y'} = x + \frac{y}{y'}$, $\frac{y'''}{y''} = x + 2\frac{y'}{y''}$, $\frac{y''''}{y'''} = x + \frac{3y''}{y'''}$, and so on. Then by substituting and by little calculation we finally get

$$y(x) = \cfrac{1}{x + \cfrac{1}{x + \cfrac{2}{x + \cfrac{3}{x + \cfrac{4}{\ddots}}}}}. \tag{3.12}$$

Setting $x = 0$ in (3.12) we get

$$y(x) = \cfrac{1}{\cfrac{1}{\cfrac{2}{\cfrac{3}{\cfrac{4}{\vdots}}}}},$$

and by taking little care of this infinite fraction we obtain that it is equal to

$$\frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdots}.$$

By Wallis' formula, this is equal to $\frac{\pi}{2}$. Finally, we therefore proved (with some care on convergence) that for all $x > 0$, we have

$$e^{\frac{x^2}{2}} \sqrt{\frac{\pi}{2}} = \left(\frac{x^1}{1} + \frac{x^3}{1 \cdot 3} + \frac{x^5}{1 \cdot 3 \cdot 5} + \cdots \right) + \cfrac{1}{x + \cfrac{1}{x + \cfrac{2}{x + \cfrac{3}{x + \cfrac{4}{\ddots}}}}}. \tag{3.13}$$

Thus, (3.13) holds also for $x = 1$. Thus, the identity (3.1) is proved. $\square$

In the Collected Papers [6] of S. Ramanujan there are many interesting theorems, identities, approximations, formulas, and conjectures. For instance, Ramanujan proved that for a sufficiently big natural number n, there are, as a rule, $\log \log n$ prime divisors of n. The next example is his approximation $\pi \approx \frac{63(17+15\sqrt{5})}{25(7+15\sqrt{5})}$ which is exactly up to 9 decimals (of course, without computers!). Even today, certain aspects of the so-called „combinatorial Rogers-Ramanujan identities "are the topic of current research, e.g. see [12] by Croatian mathematician Mirko Primc, an expert in applications of representation theory and Lie algebra theory in combinatorics.

Here is one of Ramanujan's problems posed in 1913 (from [6]):

Compute

a)

$$\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \cdots}}}, \tag{3.14}$$

b)

$$\sqrt{6 + 2\sqrt{7 + 3\sqrt{8 + \cdots}}}. \tag{3.15}$$

Solution by S. Ramanujan.
a) It holds $n(n+2) = n\sqrt{1+(n+1)(n+3)}$. Let $f(n) = n(n+2)$. Then

$$f(n) = n\sqrt{1+f(n+1)} = n\sqrt{1+(n+1)\sqrt{1+f(n+2)}} = \cdots ,$$

hence

$$n(n+2) = n\sqrt{1+(n+1)\sqrt{1+(n+2)\sqrt{1+\cdots}}}.$$

For $n = 1$, the result of a) is equal to 3.
b) Similarly, let $f(n) = n(n+3)$. Since $n(n+3) = n\sqrt{n+5+(n+1)(n+4)}$, we have

$$f(n) = n\sqrt{n+5+f(n+1)}$$
$$= n\sqrt{n+5+(n+1)\sqrt{n+6+f(n+2)}} = \cdots ,$$

and for $n = 1$, we get that the result of b) is equal to 4.
Ramanujan also conjectured many identities and a lot of claims which he or other people resolved later. One easy is that the number 0.2357111317192329... (concatenation of all primes after the decimal comma) is not a rational number. But his conjecture that the number $\pi + e$ is not rational is still not resolved. He knew Leibniz's formula $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$, which follows from the series expansion $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ for $x = 1$, and Euler's number $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$ and was not sure about their sum, although both π and e are transcendental. Also, an open problem is that 2^e is not rational. It was only in 1934 that was proved that 2 to the power of $\sqrt{2}$ and $\frac{\log 2}{\log 3}$ are transcendental. It is quite possible that Ramanujan considered such problems much earlier.
Apparently, Leibniz's formula for $\frac{\pi}{4}$ was known to the Indian mathematician Nilakanthi Somayai (1444-1530) and also to Scott James Gregory (1638-1675). Leibnitz proved it in 1673 geometrically. Even Newton praised that formula and said that it showed that Leibniz was a genius, although they had a long dispute about whose contribution to calculus was most influential. In any case, Srinivasa Ramanujan definitely was a genius.
In the end, we provide some questions, sayings, and open problems I put in [16].
Besides Millenium problems which I included in my translation (not in the original), I also added some solved and unsolved problems in the translation [16] of [11].
Here are some.

- Four girls are bathing in the (say) Adriatic sea. Each two are at a distance from each other at about 25 meters. Three of the girls have red bikinis, what has the fourth girl on herself?

- (Paul Erdös,1936) If a set A of natural numbers has the property that the sum of the reciprocals diverges, then A has an arithmetical sequence of arbitrary length (true for A primes, by T. Tao and B. Green, 2012)

- David Hilbert (1862 -1943) once said that if he awakens (in some sense) 1000 years from now, his first question would be: Is the Riemann hypothesis solved?

- Euler's perfect brick (or box) problem (about 1772): Is there a perfect brick? A perfect brick is a quadrum (brick) with all lengths of edges, diagonals (plane and space) are whole numbers.

- Graham's problem (1996): Is the sequence (a_n), unbounded if $a_0 = 2$ and $a_{n+1} = a_n - \frac{1}{a_n}$?

- Geometry problem (from 1930th): Which polyhedron on n vertices on the unit sphere has the maximal volume? (The five Plato's bodies are solutions, but in general?)

- Atiyah's conjecture (1998) on star configurations: Consider $n > 2$ points („stars ") in space, not all on a line. From any point („star ") consider $n-1$ directions to other „stars "(considered as complex numbers on the unit sphere). Attach to any point („star ") the polynomial whose directions are the roots. Then is the set of these n polynomials linearly independent (over the complex numbers)?

- (D. Veljan, 2023): The probability that a randomly and uniformly chosen point from the circumball of a tetrahedron is out of the inscribed ball is greater than or equal to $1 - \sqrt{\frac{d_3}{(3e_1)^3}}$, (see [15]) where $e_1 = aa' + bb' + cc'$, $d_3 = (aa' + bb' - cc')(aa' - bb' + cc')(-aa' + bb' + cc')$, and a and a' are the opposite side lengths of a tetrahedron and similarly for b, b' and c, c'. (We can think of vertices of the tetrahedron as stars and the chosen point as an exoplanet.) What are the hyperbolic 3D and 4D versions of this fact with respect of the complexity of our Universe?

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