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Maximal matchings in multiple ring networks with shared link

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Abstract

Graph theory is an extremely diverse field with wide applications today. Graphs have proven to be an excellent tool for modeling systems, emphasizing connections and relationships between objects. In graph theory, matching is a fundamental concept used to describe a set of edges without common vertices. Understanding them is essential for solving problems involving efficient routing and resource allocation. In this work, we enumerate maximal matchings and determine the saturation and matching number in book graphs, which are suitable for representing certain configurations of computer networks.

Keywords: cycle related graphs, book graph, maximal matching, saturation number, matching number.
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1 Introduction

Graph theory is an extremely diverse field with wide applications today. Graphs have proven to be an excellent tool for modeling systems emphasizing connections and relationships between objects. If we pay attention, we will notice that the problems studied by graph theory are everywhere around us. In this paper, we will show the properties of book graphs that are inspired by a type of network topology.

A graph $G(V, E)$ is a pair of two sets, V and E, $V = V(G)$ being a finite nonempty set and $E = E(G)$ is a binary relation defined on V. A graph can be visualized by representing the elements of V by points (vertices) and

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joining pairs of vertices (i, j) by an edge (bond) if and only if $(i, j) \in E(G)$. The number of vertices in G equals the cardinality $n = |V(G)|$ of this set. The degree of a node in a non-directed graph is defined as the number of links a node has with other nodes.
We assume that the reader is familiar with basic graph-theoretic concepts such as degree, neighborhood, etc., and with basic classes of graphs such as paths, cycles, and complete graphs. Here we denote by P_n a path on n edges ($n+1$ vertices) and by C_n a cycle on n vertices. All graphs in this paper are finite, simple, and undirected. Terms not defined here are used in the sense of Harary [4].

In graph theory, *matching* is a fundamental concept used to describe a set of edges without common vertices. Matchings are used in various applications such as network design (efficient routing and resource allocation), job assignments (assigning jobs to machines or workers), scheduling (optimal scheduling of tasks), chemistry, graph coloring, neural networks in artificial intelligence, and more. The cardinality of M is called the size of the matching. As the matchings of small sizes are not interesting, we will be mostly interested in matchings that are as large as possible. A matching M is *maximum* if there is no matching in G with more edges than M. The cardinality of any maximum matching in G is called the *matching number* of G and denoted by $\nu(G)$. Since each vertex can be incident to at most one edge of a matching, it follows that the matching number of a graph on n vertices cannot exceed $\left\lfloor \frac{n}{2} \right\rfloor$.
The matching M is *perfect* if each vertex of G is incident with an edge of M. Perfect matchings (also known in chemistry as Kekulé structures) are also maximum matchings [5].
There is another way to quantify the idea of ”large” matching. A matching M in G is *maximal* if no other matching in G contains it as a proper subset. Obviously, every maximum matching is also maximal, but the opposite is generally not true. Maximal matchings are much less researched with respect to both their structural and enumerative properties. Maximal matchings can serve as models of several technical problems such as the block-allocation of a sequential resource. The cardinality of any smallest maximal matching in G is the *saturation number* of G. The saturation number of a graph G we denote by $s(G)$. It is easy to see that the saturation number of a graph G is at least one-half of the matching number of G, i.e., $s(G) \geq \nu(G)/2$. Hence, the saturation number provides a piece of information on the worst possible case.

Network topology refers to the arrangement and interconnection of various components within a (computer) network, including nodes (computers,

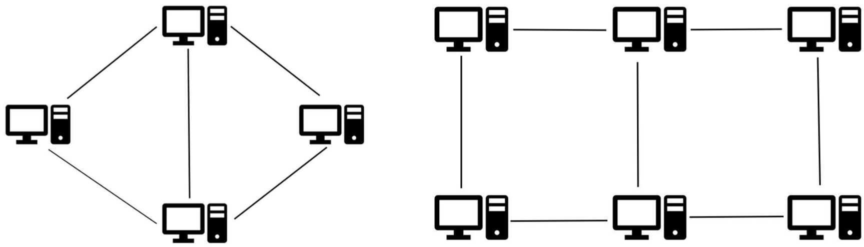

Figure 1: Multiple Ring Networks with Shared Link

switches, routers) and links (wired or wireless connections). It defines how these components are connected and interact with each other. Physical topology refers to the placement of the network's various components, including the device locations and cable installation, while logical topology shows how data flows within the network, regardless of its physical design [3].
The structure of a network topology determines how data is transmitted, affecting the network's performance, reliability, and scalability. An efficiently designed topology can reduce cable costs, enhance data transfer speeds, and improve network reliability. On the other hand, a poorly thought-out topology can lead to congested data paths and increased risk of network failures. For organizations, choosing the right topology is a key part of network planning, as it affects both the operational efficiency and the ease of future expansion. The landscape of network topology is diverse, offering various configurations, each with its unique characteristics and suitability for different network scenarios.

The primary types of network topology include: Point-to-Point (represented by path graph), Bus Topology (caterpillar), Star (star graph), Ring (cycle graph), Tree (tree graph), Mesh (with each node having a connection to several other nodes), Hybrid (combines two or more different types of topologies).
In this paper, we are concerned with graphs representing one possible network topology, the Multiple Ring Network with Shared Link. The shape of that topology can be represented by a graph we call a book graph. We represent the components, nodes (computers, switches, routers) by vertices and links (wired or wireless connections) by edges of certain graphs. Book graphs consist of a certain number of cycles, not necessarily of the same length, which all share one common edge. The cycle lengths are at least three. See examples in Fig.2.

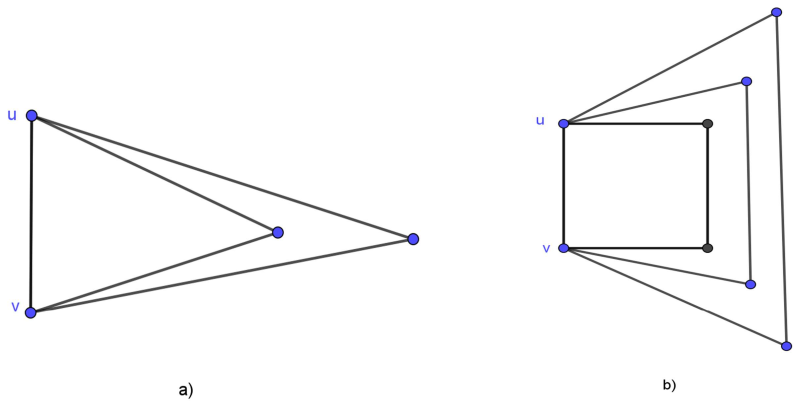

Figure 2: A book graph a) $B(2,1)$ with 2 sheets b) $B(3,2)$ with 3 sheets

2 Maximal matchings in book graphs

In this section, we state and prove our main results about the number of maximal matchings in book graphs. We refer the reader to [4] for all graph-theoretical terms not defined here.

A *book graph* $B = B(n,k)$ is a graph with $nk+2$ vertices, consisting of n cycles C_{k+2}, that all share exactly one common edge. Let us denote the vertices of the common edge with u and w. The other vertices we denote with $v_{11},...,v_{1k},v_{21},...,v_{2k},...v_{n1},...,v_{nk}$, where the first label, say m, indicates the cycle, and the second label indicates the position of a vertex in the m-th cycle. In all cycles, the second vertex labels are increasing when proceeding along the cycle from u to w. See Figure 3. In order to avoid problems with too few, or with too short cycles, we restrict our attention to $n \geq 2$ and $k \geq 3$. We denote the number of maximal matchings in $B(n,k)$ by $\Psi(n,k)$.
First, we settle the two shortest cases, $k=1$ and $k=2$.

Lemma 2.1. *Let $n \geq 2$. Then the number of maximal matchings in $B(n,1)$ is given by*

$$\Psi(n,1) = n(n-1)+1, \tag{2.1}$$

and the number of maximal matchings in $B(n,2)$ is given as

$$\Psi(n,2) = n^2+1. \tag{2.2}$$

Proof. We start with an observation, valid also for $k \geq 3$, that any maximal matching must cover at least one of the vertices u and w. The number of

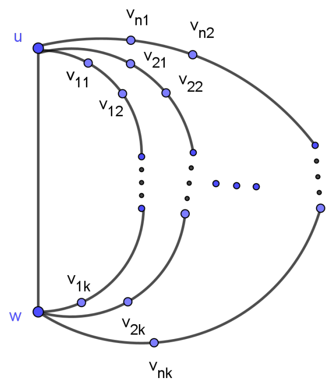

Figure 3: A book graph $B(n, k)$

maximal matchings in $B(n, 1)$ in which both u and w are covered by the same edge, hence uw, is exactly one. If u and w are covered by different edges, the edge covering u can be chosen in n ways, leaving $n-1$ possibilities to choose the edge covering w, since those edges cannot belong to the same cycle. Since it is not possible to have either one of u and w uncovered by a maximal matching, we have exhausted all possibilities and the total number of maximal matchings in $B(n, 1)$ is equal to $n(n-1)+1$, as claimed.
The number of maximal matchings in $B(n, 2)$ covering both u and w by the same edge is again 1. If those vertices are covered by different edges, there are $n \cdot n = n^2$ such possibilities. Again, it is not possible to have just one of them covered by a maximal matching, a consequence of random matchability of cycles C_4 making the sheets of the considered books. Hence, $\Psi(n, 2) = n^2 + 1$, as claimed in the statement. □

Both sequences $\Psi(n, 1)$ and $\Psi(n, 2)$ appear in the *On-Line Encyclopedia of Integer Sequences* [1], $\Psi(n, 1)$ as A002522, and $\Psi(n, 2)$ as A002061. Both have a number of other combinatorial interpretations, but maximal matchings are not among them. It would be an interesting exercise to construct explicit bijections between some of those interpretations and our maximal matchings.

Before we consider the general case, we notice for that any maximal matching in $B(n, k)$ covering both u and w, the remaining graph decomposes into a disjoint union of paths of the same, or almost same, length. Hence, we quote a result on the number of maximal matchings in paths [2].

Proposition 2.2 ([2]). *Let ψ_k denote the number of maximal matchings in P_k. The sequence ψ_k satisfies the linear recurrence*

$$\psi_k = \psi_{k-2} + \psi_{k-3},$$

with the initial conditions $\psi_0 = \psi_1 = \psi_2 = 1$.

The enumerating sequence of the number of maximal matchings in P_k is the (shifted) Padovan sequence, sequence A000931 from [1]. We invite the reader to refer to OEIS for other combinatorial representation of this sequence.
The above lemma will be useful also for the case when only one of u and w is covered by a maximal matching. In that case, all neighbors of the other one must be covered, and the graph again decomposes into several disjoint paths, their length again quite similar.

Proposition 2.3. *The sequence $\Psi(n,k)$ is given by*

$$\Psi(n,k) = \psi_k^n + n\,\psi_{k-2}\,\psi_k^{n-1} + n(n-1)\,\psi_{k-1}^2\,\psi_k^{n-2} + 2\,n(n-1)\,\psi_{k-3}\,\psi_{k-2}^{n-1},$$

for $n \geq 2, k \geq 3$, where ψ_k denotes the number of maximal matchings in P_k.

Proof. As mentioned before, any maximal matching in $B(n,k)$ must cover at least one of the vertices u and w. We first look at the case when it covers both of these vertices. The number of maximal matchings covering them with the edge uw is the number of maximal matchings in $B(n,k)\backslash\{u,w\}$. This graph is a disjoint union of n paths P_k, each of them has ψ_k maximal matchings. Therefore, the number of maximal matchings covering vertices u and w by the same edge is equal to

$$\psi_k^n. \tag{2.3}$$

Let us now consider the case when u and w are covered by different edges in a maximal matching. If both of these edges are in the same cycle, say uv_{l1} and wv_{lk}, what remains when we remove them, is a disjoint union of $n-1$ copies of P_k and one copy of P_{k-2}. The number of maximal matchings in that case is $\psi_k^{n-1}\psi_{k-2}$. Since there are n possibilities for choosing the cycle in which the edges covering u and w are, the total number of such maximal matchings is

$$n\,\psi_{k-2}\,\psi_k^{n-1}. \tag{2.4}$$

In the same way, we can conclude that the number of maximal matchings in which u and w are covered by edges from different cycles is equal to

$$n(n-1)\,\psi_{k-1}^2\,\psi_k^{n-2}. \tag{2.5}$$

The last possible case is that the maximal matching covers only one of the vertices u and w, say u. Let it be covered by the edge uv_l. Then all the neighboring vertices of w must be covered by edges $v_{1k}v_{1,k-1}, ..., v_{nk}v_{n,k-1}$. So, we have one copy of P_{k-3} and $n-1$ copies of P_{k-2}, with n such possible situations, so we have $n\psi_{k-3}\psi_{k-2}^{n-1}$ maximal matchings that cover only u. By symmetry, there are exactly as many maximal matchings that cover only w, so the number of maximal matchings that cover only one of the vertices u and w is equal to

$$2n\,\psi_{k-3}\,\psi_{k-2}^{n-1}. \tag{2.6}$$

Now we get the total number of maximal matchings in $B(n,k)$ for $n \geq 2, k \geq 3$ by summing all possible cases.

□

For the same reason, as in the proof for maximal matchings, we state the result for saturation number, where n denotes the number of vertices [2],

$$s(P_n) = \left\lfloor \frac{n+1}{3} \right\rfloor \tag{2.7}$$

and matching number for paths

$$\nu(G) = \left\lfloor \frac{n}{2} \right\rfloor. \tag{2.8}$$

Proposition 2.4. *Saturation number for graph $B(n,k)$ is equal to*

$$s(B(n,k)) = \begin{cases} (n-1)\left\lfloor \frac{k-1}{3} \right\rfloor + \left\lfloor \frac{k-2}{3} \right\rfloor, & \text{if } k \geq 3 \\ 2\left\lfloor \frac{k}{3} \right\rfloor + (n-2)\left\lfloor \frac{k+1}{3} \right\rfloor, & \text{if } k = 2. \end{cases}$$

Proof. Let's consider all possible cases. Any maximal matching in $B(n,k)$ must cover at least one of the vertices u and w.

We first look at the case when it covers both of these vertices. This graph is a disjoint union of n paths P_k, and each of them has saturation number $s(P_k) = \left\lfloor \frac{k+1}{3} \right\rfloor$. Therefore, saturation number in this case is equal to

$$n\left\lfloor \frac{k+1}{3} \right\rfloor + 1 \tag{2.9}$$

In the case when u and w are covered by different edges we have two options. If both of these edges are in the same cycle, for the disjoint union of $n-1$ copies of P_k and one copy of P_{k-2} saturation number is equal

$$(n-1)\left\lfloor\frac{k+1}{3}\right\rfloor+\left\lfloor\frac{k-1}{3}\right\rfloor \tag{2.10}$$

In the same way, we can conclude that the saturation number in case when u and w are covered by edges from different cycles is equal to

$$2\left\lfloor\frac{k}{3}\right\rfloor+(n-2)\left\lfloor\frac{k+1}{3}\right\rfloor \tag{2.11}$$

If the maximal matching covers only one of the vertices u and w, then we have one copy of P_{k-3} and $n-1$ copies of P_{k-2}, so the saturation number is equal to

$$(n-1)\left\lfloor\frac{k-1}{3}\right\rfloor+\left\lfloor\frac{k-2}{3}\right\rfloor, \tag{2.12}$$

which is also the smallest possible value for saturation number. For $k=2$ formulas (11) and (12) give the same value.

□

By analogical consideration it can be easily shown that the formula for matching number in $B(n,k)$ is given by the following proposition.

Proposition 2.5.

$$\nu(B(n,k))=n\left\lfloor\frac{k}{2}\right\rfloor+1.$$

Proof. In the first case when maximal matching covers both of joint vertices, and graph is a disjoint union of n paths P_k, is also maximum matching in $B(n,k)$.

□

3 Concluding remarks

In this paper we have enumerated maximal matchings in a class of cycle related graphs, interesting from the viewpoint of topology of computer networks. Our results could be generalized in a straightforward way to similar network configurations, in particular to multiple ring networks sharing a single node, and we leave it to the interested reader. For both types of networks, several interesting problems remain unanswered. Of particular interest would be to compare the results for two cycles with a larger number of vertices compared to cases with multiple cycles with a smaller number of vertices. Some preliminary investigations are underway and we hope to be able to report conclusive results soon.

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