2nd Croatian Conference on Geometry and Graphics
Šibenik, September 5–9, 2010

ABSTRACTS

EDITORS:
Tomislav Došlić, Marija Šimić

PUBLISHER:
Croatian Society for Geometry and Graphics

Supported by the Ministry of Science, Education and Sports of the Republic of Croatia.

Contents

Plenary lectures

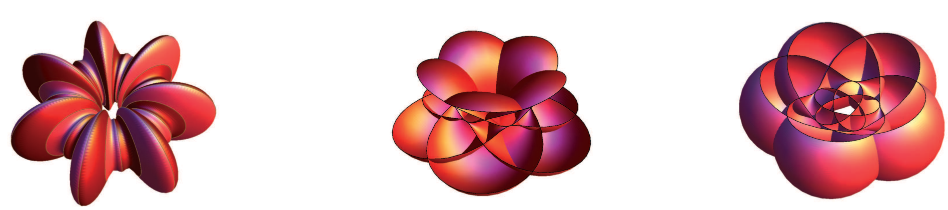
Circular surfaces $\mathcal{CS}(\alpha, p)$

Sonja Gorjanc
Faculty of Civil Engineering, University of Zagreb, Zagreb, Croatia
e-mail: sgorjanc@grad.hr

This lecture introduces a new concept of surface-construction: We consider a congruence of circles $\mathcal{C}(P_1, P_2) = \mathcal{C}(p)$ in the Euclidean space, i.e. a two-parametric set of circles which pass through the points P_1, P_2 given by the coordinates $(0, 0, \pm p)$, where $p = \sqrt{q}$, $q \in \mathbb{R}$. It is a normal curve congruence with singular points on the z axes, [3]. $\mathcal{C}(p)$ is a hyperbolic, parabolic or elliptic if q is greater, equal or less then 0, respectively. For a piecewise-differentiable curve $\alpha : I \to \mathbb{R}^3$, $I \subset \mathbb{R}$, we define a *circular surface* $\mathcal{CS}(\alpha, p)$ as the system of circles of $\mathcal{C}(p)$ which cut the curve α.

For the surfaces $\mathcal{CS}(\alpha, p)$ we derive parametric equations (which enable their visualizations in the program *Mathematica*) and investigate their properties if α is an algebraic curve. In the general case, if α is an algebraic curve of the order n, $\mathcal{CS}(\alpha, p)$ is an algebraic surface of the order $3n$ passing n times through the absolute conic and containing the n-fold straight line P_1P_2. But the order of $\mathcal{CS}(\alpha, p)$ is reduced if α passes through the absolute points or if it cuts the line P_1P_2.

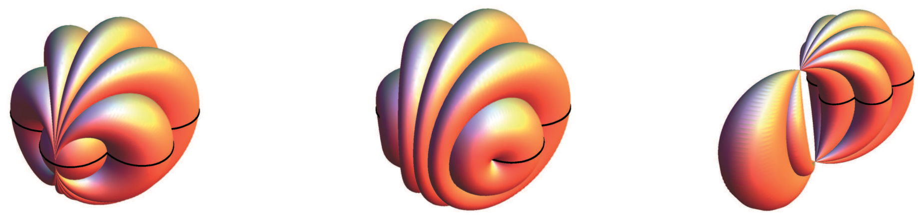
The first examples of algebraic $\mathcal{CS}(\alpha, p)$ are parabolic cyclides (if α is a line), Dupin's cyclides (if α is a circle) and rose-surfaces (if α is a rose) [1], [4]. Furthermore, we consider cyclic-harmonic curves $R(a, n, d)$ lying in the plane $z = k$, $k \in \mathbb{R}$, which are given by the polar equation $\rho = \cos(\frac{n}{d}\varphi) + a$, where $\frac{n}{d}$ is a positive rational number in lowest terms and $a \in \mathbb{R}^+ \cup \{0\}$. A *generalized rose-surface* $\mathcal{R}(p, k, n, d, a)$ is the surface $\mathcal{CS}(\alpha, p)$ where the directing curve α is the cyclic harmonic curve $R(n, d, a)$ in the plane $z = k$. These surfaces have various attractive shapes, a small number of high singularities, and they are convenient for algebraic treatment and visualization in the program *Mathematica*. Some examples are shown in Fig. 1.



a b c

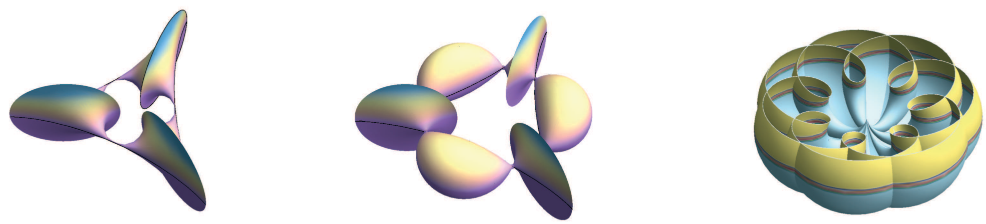
Figure 1: $\mathcal{R}(i, 0.75, 7, 1, 2)$ in Fig. a, and two different cuts of $\mathcal{R}(i, 0, 5, 3, 2)$ in Figs. b and c.

Since the surface-construction concept mentioned above can be applied on any curve α, numerous new forms of surfaces can be obtained. Some examples are shown in Fig. 2 and Fig. 3.



a b c

Figure 2: The surfaces directed by a parabolic, elliptic and hyperbolic congruence and one cyclic harmonic curve are shown in Figs a, b and c, respectively.



a b c

Figure 3: The surfaces directed by Steiner curve (hypocycloid) in the plane $z = 0$, and $p = i, 1.5i$ are shown in figs. a and b, respectively. The surface in fig. c is directed by $p = 0$ and one epitrochoid in the plane $z = 0$.

Key words: circular surfaces, congruence of circles, cyclic-harmonic curves, generalized rose-surfaces

MSC 2010: 51N20, 51M15

References

[1] S. Gorjanc, Rose Surfaces and their Visualizations, *Journal for Geometry and Graphics* **14** (1) (2010) 59–67.

[2] H. Hilton, On Cyclic-Harmonic Curves, *The Annals of Mathematics*, Second Series **24** (3) (1923) 209–212.
`http://www.jstor.org/stable/1967850`

[3] G. Salmon, *A Treatise on the Analytic Geometry of Three Dimensions*, Vol.II., Chelsea Publishing Company, New York, 1965 (reprint).

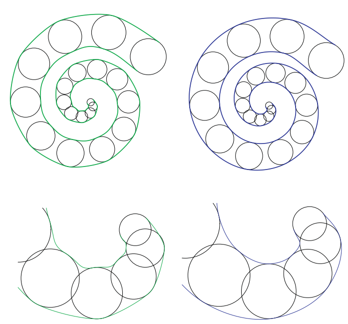
[4] "Special Rose Surfaces" from The Wolfram Demonstrations Project
`http://demonstrations.wolfram.com/SpecialRoseSurfaces/`
Contributed by: Sonja Gorjanc

Skinning of circles by Hermite interpolation and biarcs

Miklós Hoffmann
Institute of Mathematics and Informatics, Eszterházy Károly University, Eger, Hungary
e-mail: hofi@ektf.hu

Roland Kunkli
Department of Computer Graphics and Image Processing, University of Debrecen, Debrecen, Hungary
e-mail: rkunkli@inf.unideb.hu

Special interpolation of an ordered set of discrete circles or spheres is discussed in this presentation, which is frequently referred to as skinning in Computer Aided Geometric Design. By skinning in 2D we mean the geometric construction of two curves touching each of the given circles. Visually satisfactory result is required, i.e. curves or surface without unnecessary oscillations, bumps and loops. Here we precisely define the admissible set of circles and the desired curve. For any admissible data we can create the G^1 continuous skins, finding the touching points by Apollonius circles. To create the final shape we use Hermite interpolation, and also biarcs, that is curves made by joining two circular arcs. Results of the presented method (right in the Figure) are compared to skins obtained by the recent numerical technique of Slabaugh (left).



Key words: skinning, circles, Hermite interpolation, Apollonius circles, biarcs

MSC 2010: 68U05

References

[1] M. Peternell, B. Odehnal, M.L. Sampoli, On quadratic two-parameter families of spheres and their envelopes, *Computer Aided Geometric Design* **25** (2008) 342–355.

[2] G. Slabaugh, G. Unal, T. Fang, J. Rossignac, B. Whited, Variational Skinning of an Ordered Set of Discrete 2D Balls, *Lecture Notes on Computer Science* **4795** (2008) 450–461.

[3] M. Hoffmann, R. Kunkli, Skinning of circles and spheres, *Computer Aided Geometric Design* (submitted).

Geometry – science, art or game

Ana Sliepčević
Faculty of Civil Engineering, University of Zagreb, Zagreb, Croatia
e-mail: anas@grad.hr

The research within the area of the curve theory has been moving in at least two directions, one being constructive method, and the other analytic. Both ways are scientifically sufficient and have the same base – *synthetic method*.

The phrase *curve* necessarily includes its visualization, these days obtained by both mentioned research methods. In my opinion, the synthetic-constructive method is more convenient to many geometers since it elegantly leads to the desired results, without use of coordinates and systems of equations.

Without claiming to show something new and original here we give an overview of a large set of the plane rational curves and envelopes associated with the conics in different ways. The aim is to emphasize the *beauty of geometry* on one hand, and the *simplicity and elegance of the synthetic method* leading to the desired results on the other. It is necessary to have only one of many offered dynamic computer programs, sufficient knowledge of the synthetic geometry, and a little bit of imagination. The walk starts through the ocean of the curves associated with the conics in the Euclidean plane, and it continues in the pseudo-Euclidean and hyperbolic plane by using the Cayley-Klein models. It is easy to perceive the behavior of the curves in the real absolute points, which in the Euclidean case, because of the imaginarity of the absolute points, is often difficult even to visualize. Geometry is a science. But at the end, I expect you to agree with me that it is an art as well. Once when you become enough possessed by it, it becomes an intellectual game.

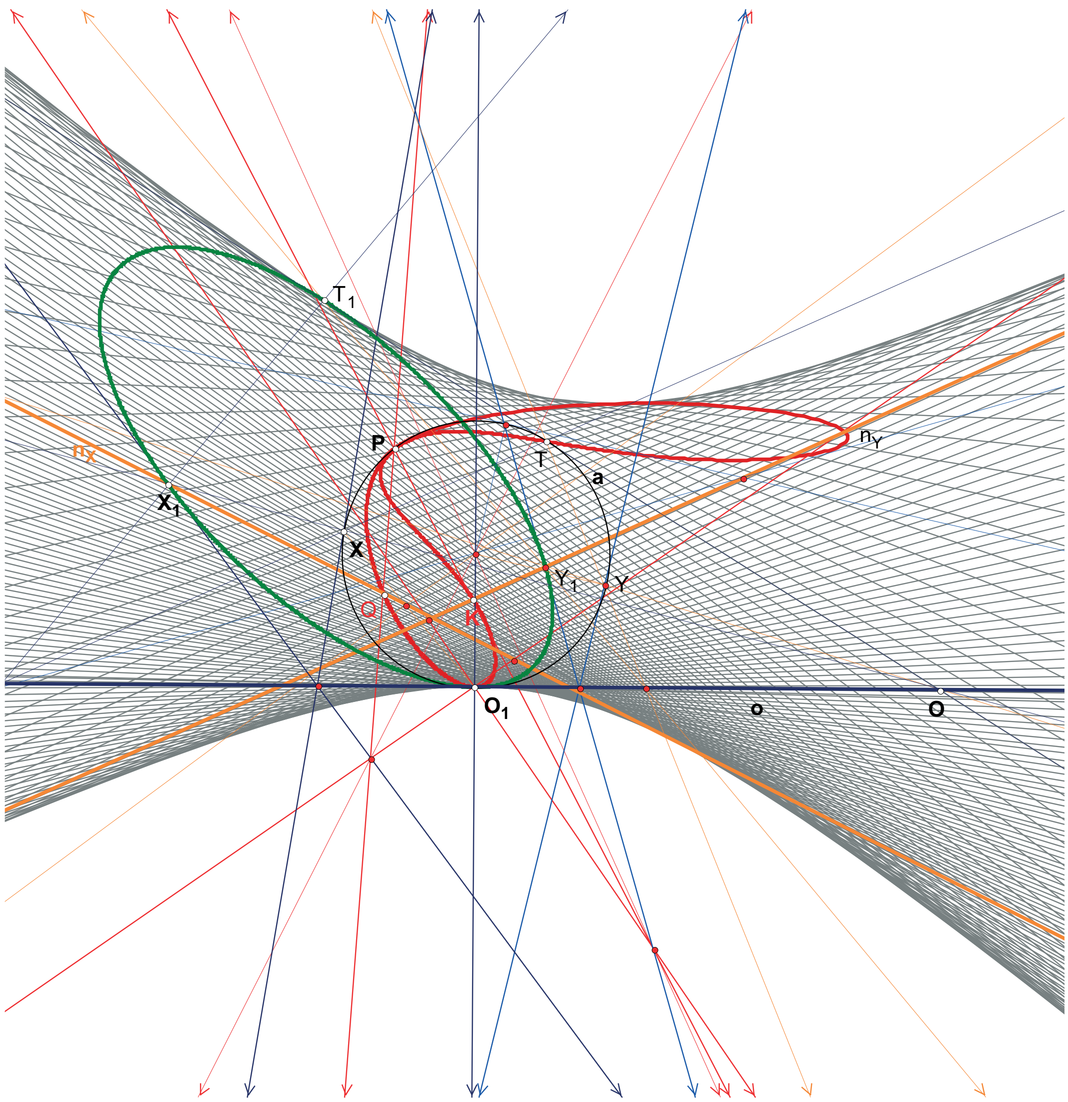


Figure 1: The evolute of a special hyperbola and its pedal quartic curve.

Key words: synthetic geometry, pseudo-Euclidean plane, hyperbolic plane, curve, evolute

MSC 2010: 51M15, 51M19

References

[1] A.A.Savelov, *Ravninske krivulje*, Školska knjiga, Zagreb, 1979.

[2] H. Wieleitner, *Theorie der ebenen algebraischen Kurven höherer Ordnung*, G. J. Göschen'sche Verlagshandlung, Leipzig, 1905.

Rotor coordinates, vector trigonometry and Kepler-Newton orbits

Norman John Wildberger
University of New South Wales, Sydney, Australia
e-mail: n.wildberger@unsw.edu.au

We study a new, yet old, rotor coordinate system for vectors in the plane, yielding a trigonometry ideally suited for engineering, physics and graphics applications. This 'vector trigonometry' lies somewhere between classical trigonometry and rational trigonometry.

Rotor coordinates allow us to see many computations of classical geometry in a different light. We give some applications to quadrilateral formulas, and to a new look at Newton's derivation of Kepler's law of planetary motion.

Key words: rotor coordinates, vector trigonometry, planar kinematics, Newton Kepler orbits

MSC 2010: 51Kxx, 70Mxx

Contributed talks

Towards classification of conic pencils in pseudo-Euclidean plane

Jelena Beban-Brkić
Faculty of Geodesy, University of Zagreb, Zagreb, Croatia
e-mail: jbeban@geof.hr

Marija Šimić
Faculty of Architecture, University of Zagreb, Zagreb, Croatia
e-mail: marija.simic@arhitekt.hr

Pseudo-Euclidean plane is a real affine plane where a metric is induced by an absolute figure $(\omega, \Omega_1, \Omega_2)$ consisting of the line at infinity ω and points $\Omega_1, \Omega_2 \in \omega$. The classification of conics in pseudo-Euclidean plane was carried out in [5]. In this lecture we determine the properties of conic pencils of type I with four real and distinct fundamental points. The notion of pseudo-orthogonal matrix concerning pseudo-Euclidean motions is introduced as well.

Key words: pseudo-Euclidean plane, conic section, fundamental points, conic pencils of type I

MSC 2010: 51A05, 51N25

References

[1] D. Blanuša, *Viša matematika*, I dio, Tehnička knjiga, Zagreb, 1965.

[2] J. Beban-Brkić, Isometric invariants of conics in the isotropic plane-classification of conics, *Journal for Geometry and Graphics* **6** (1) (2002) 17–26.

[3] B.Divjak, *Geometrija pseudogalilejevih prostora*, PMF-Matematički odjel, Sveučilište u Zagrebu, doktorska disertacija, Zagreb, 1997.

[4] I. M. Yaglom, *Princip otnositeljnosti Galileja i neevklidova geometrija*, Nauka, Moskva, 1969.

[5] N. V. Reveruk, Krivie vtorogo porjadka v psevdoevklidovoi geometrii, *Uchenye zapiski Moskovskogo pedagogicheskogo instituta* **253** (1969) 160–177.

[6] V. G. Shervatov, *Hyperbolic functions*, Dover Publications, New York, 2007.

[7] V. Šćurić - Čudovan, Zur Klassifikationstheorie der Kegelschnittbüschel der isotropen Ebene, I. Teil, *Rad JAZU* **450** (1990) 41–51.

Six–point perspectives of different kinds

Attila Bölcskei
Miklós Ybl Faculty of Architecture and Civil Engineering, Szent István University, Budapest, Hungary
e-mail: bolcskei.attila@ybl.szie.hu

Perspective representation is in the focus of education of architects and in graphic arts. The standard teaching material usually contains the one-point, two-point and possibly the three-point perspective systems. It is also well-known that curvilinear perspective has been developed to approximate the eyes representation more accurately. The four-point perspective can be considered the equivalent of two-point perspective, while five-point perspective (also mentioned as fish-eye perspective), where four vanishing points are placed around in a circle and the remaining one vanishing point in the center of the circle, is the curvilinear equivalent of one-point perspective.

In this talk we intend to introduce the efforts made to represent the last (sixth) vanishing point on a surface or in the plane. We also show elementary constructions that remain valid in a family of six-point perspectives.

Key words: curvilinear descriptive geometry, perspective

MSC 2010: 51N05, 65D17

References

[1] D. Lörincz, Á. Urbin, B. Szilágyi, *Space Representation with Six Vanishing Points*, talk at the conference Projecting Spaces, Internationale Konferenz zur Architekturdarstellung an der BTU, Cottbus, 2009.

[2] D.A. Termes, Six-Point Perspective on the Sphere: The Termesphere, *Leonardo* **24** (3) (1991) 289–292.

Osculating circles of conics

Ivana Božić
Department of Civil Engineering, Polytechnic of Zagreb, Zagreb, Croatia
e-mail: ivana.bozic@tvz.hr

In the Euclidean plane there are several well-known methods of constructing an osculating (Euclidean) circle to a conic. I will show general constructive methods of an osculating circle at a point of a conic given with five points. These methods can be "translated" into a construction scheme of finding the osculating circle to a given conic in a hyperbolic or elliptic plane.

Key words: elation, pencil of conics, osculating circle, curvature center

MSC 2010: 51M15, 51N05

References

[1] G. Weiss, A. Sliepčević, Osculating circles of conics in Cayley-Klein Planes, *KoG* **13** (2009) 7–12.

Contemporary principles of geometrical modeling in education

Aleksandar Čučaković
Faculty of Civil Engineering, University of Belgrade, Belgrade, Serbia
e-mail: cucak@grf.bg.ac.rs

Miodrag Nestorović
Faculty of Architecture, University of Belgrade, Belgrade, Serbia
e-mail: enestorm@arh.bg.ac.rs

Biljana Jović
Faculty of Forestry, University of Belgrade, Belgrade, Serbia
e-mail: ngbilja@arfodita.rcub.bg.ac.rs

This paper deals with changes in forms of geometrical education and directions of transforming of the subject Descriptive Geometry and its adaptation to contemporary conditions. Following presentation is an overview of the direct application of geometric education in the Construction Systems and Space Structures - Metamorphosis of the space.

The possibility of viewing the objects from all sides and research in space is an interactive and dynamic component offering the excellent training of user's spatial abilities. This type of education in VR environment is a new potential for the study of geometry. The construction of 3D objects in 3D space by VR in accordance with pedagogical theories created a process that supports users in developing the capacity of spatial ability. The 3D dynamic geometry is impossible to realize with traditional conventional methods of study as well as with the existing CAD programs.

Because of the largeness of the material covered by the area of Descriptive Geometry and great possibilities of modern technology, it is important to measure properly the size and position of individual courses that are taught on technical universities, with the necessity of preserving the theoretical knowledge of descriptive geometry.

Key words: geometry, education, modeling, metamorphoses, VR

MSC 2010: 51N05

Authors are supported by the Ministry of Science and Technological Development, Republic of Serbia, Project No. 16009A.

References

[1] A. Čučaković, *Nacrtna geometrija*, Akademska misao, Beograd, 2010.

[2] A. Čučaković, M. Dimitrijević, Opšti i posebni nastavni sadržaji u edukaciji u Nacrtnoj geometriji i inženjerskoj grafici, in *Conference proceedings "moNGeometrija 2006"*, Novi Sad, Serbia, 2006, 199 – 210.

[3] A. Čučaković, B. Jović, *Optional course Engineering Graphics on Department for Landscaping Architecture at the Faculty of Forestry, University of Belgrade*, at International Conference moNGeometrija 2010, Belgrade, Serbia, June 24-27, 2010.

[4] H. Kaufmann, *Geometry education with augmented reality*, PhD thesis, TU Vienna, 2004.

[5] M. Mitić, B. Jović, A. Čučaković, *Predlozi za unapređenje primjene nacrtne geometrije u nastavnom procesu na Odseku za pejzažnu arhitekturu Šumarskog fakulteta Univerziteta u Beogradu*, at International Conference moNGeometrija 2008, Niš, Serbia, September 25-27, 2008.

[6] H. Stachel, *On the role of Descriptive Geometry in the different curricula*, at International Conference moNGeometrija 2010, Belgrade, Serbia, June 24-27, 2010.

[7] H. Stachel, *The status of today's descriptive geometry related education (CAD/CG/DG) in Europe*, at 40th anniversary of the Japan Society for Graphic Science, Tokio, Japan, 2007.

A Voronoi game in the unit disk

Tomislav Došlić
Faculty of Civil Engineering, University of Zagreb, Zagreb, Croatia
e-mail: doslic@grad.hr

We consider a two player game in which the first player chooses two points in the unit disk, and then the second player chooses his two points. The points cannot be reused, but can be arbitrarily close to the already used points. The Voronoi diagram of the four points is constructed and each player wins the total area of all cells belonging to his points. The winner is the player with the larger total area. We show that the second player always has a winning strategy and determine a lower bound on the margin of the victory.

Key words: Voronoi game, Voronoi diagram, competitive facility location

MSC 2010: 90B85, 91A05

References

[1] S. P. Fekete, H. Meijer, The one-round Voronoi game replayed, *Computational Geometry: Theory and Applications* **30** (2005) 81–94.

[2] H-K. Ahn, S-W. Cheng, O. Cheong, M. Golin, R. van Oostrum, Competitive facility location: the Voronoi game, *Theoretical Computer Science* **310** (2004) 457–467.

Examples of good practice in primary school education with respect to discovering geometric shapes

Margita Pavleković, Ivana Đurđević, Ana Perić
Faculty of Teacher Education, Josip Juraj Strossmayer University, Osijek, Croatia
e-mail: pavlekovic@ufos.hr, idjurdjevic@ufos.hr, aperic@ufos.hr

The paper presents a cooperation interface between students of the Faculty of Teacher Education and ten-year olds referring to mastering contents in the field of geometry by means of modern information technologies.

A model for extracurricular activity participation of pupils with special interest in mathematics, called "Little School of Mathematics", was established at the Faculty of Teacher Education in Osijek. Participants in the programme are ten-year-old children from 15 primary schools in Osijek. Under the supervision of their mentors, in the "Little School of Mathematics" teacher education students gain special competencies and skills in teaching geometric shapes by means of educational simulations "Fido's flower bed", "3D and orthographic views" and "Quilting bee".

Pupils use educational simulation "Fido's flower bed" to research forms of finding the area and perimeter of a rectangle.

Simulation "3D and orthographic views" is used to develop spatial sense with fourth-grade pupils. We started from application "Who wants to be an architect?". Pupils are intuitively introduced to the ideas of ground plan, vertical and lateral projection; they recognized three-dimensional shapes in various positions, sketch three-dimensional shapes consisting of cubes and their two-dimensional counterparts and notice how various 3D shapes can have the same 2D representations.

Educational simulation "Quilting bee" is used for perceiving symmetries from pupils' environment as well as for the composition of symmetries.

Key words: 2D shape, 3D shape, mathematical discovery, geometry classes

MSC 2010: 97D40

References

[1] B. Divjak (Ed.), *Ishodi učenja u visokom školstvu*, Varaždin, TIVA, FOI, 2009.

[2] L. Goljevački, K. Moguš, Little school of mathematics (Mala matematička škola), in *Proceedings of the second congress of mathematics teachers*, Zagreb, Croatian Mathematical Society, 2004, 150–151.

[3] M. Pavleković, *Matematika i nadareni učenici – Razvoj kurikula na učiteljskim studijima za prepoznavanje, izobrazbu i podršku darovitih učenika*, Element, Zagreb, 2009.

[4] M. Pavleković, Z. Kolar-Begović, Teachers contribution to the modernization of teaching mathematics, in *Proceedings of the International Scientific Colloquium on Contemporary Teaching in Osijek*, Faculty of Education, Osijek, 2003, 100–108.

[5] M. Pavleković, M. Zekić-Sušac, I. Đurđević, A novel way for detecting children's mathematical gift by using the estimates of teachers, psychologists, expert systems, and students, *International Journal of Research in Education* **1** (1) (2009) 13–30.

[6] M. Pavleković, M. Zekić-Sušac, I. Đurđević, Expert system for detecting a child's gift in mathematics, in *International Scientific Colloquium Mathematics and children (How to teach and learn mathematics)*, Josip Juraj Strossmayer University, Osijek, 2007, 98–116.

[7] E. Saito, H. Imansyah, I. Kubok, S. Hendayana, A study of the partnership between schools and universities to improve science and mathematics education in Indonesia, *International Journal of Educational Development* **27** (1) (2009) 194–204.

[8] ExploreLearning, Gizmos! Online simulations that power inquiry and understanding. http://www.explorelearning.com/ (Accessed: 7 July 2010)

Geodesics and geodesic spheres in $\widetilde{\mathbf{SL}(2,\mathbb{R})}$ geometry

Blaženka Divjak, Zlatko Erjavec
Faculty of Organization and Informatics, University of Zagreb, Varaždin, Croatia
e-mail: blazenka.divjak@foi.hr, zlatko.erjavec@foi.h

Barnabás Szabolcs, Brigitta Szilágyi
Department of Geometry, Budapest University of Technology and Economics, Budapest, Hungary
e-mail:szabolcs@math.bme.hu, szilagyi@math.bme.hu

$\widetilde{SL(2,\mathbb{R})}$ geometry is one of the eight homogeneous Thurston 3-geometries:

$$E^3,\ S^3,\ H^3,\ S^2\times\mathbb{R},\ H^2\times\mathbb{R},\ \widetilde{SL(2,\mathbb{R})},\ Nil,\ Sol.$$

The geometry of $\widetilde{SL(2,\mathbb{R})}$ arises naturally as the geometry of a fibre line bundle over a hyperbolic base plane $\mathbb{H}^2$. It is similar to Nil geometry in a sense, that Nil is a nontrivial fibre line bundle over Euclidean plane and $\widetilde{SL(2,\mathbb{R})}$ is a twisted bundle over $\mathbb{H}^2$.

In $\widetilde{SL(2,\mathbb{R})}$, we can define the infinitesimal arc length square using the method of Lie algebras. However, by the help of a projective spherical model of homogeneous Riemann 3-manifolds proposed by E. Molnar, the definition can be formulated in a more straightforward way. The advantage of this approach lies in the fact that we get a unified, geometrical model of that sort of spaces.

We give a description of the hyperboloid model of $\widetilde{SL(2,\mathbb{R})}$ geometry and discuss the geodesics. We also determine and explain the geodesic spheres.

Key words: $\widetilde{SL(2,\mathbb{R})}$ geometry, geodesics, geodesic sphere

MSC 2010: 53A35, 53C30

References

[1] A. Bölcskei, B. Szilágyi, Frenet formulas and geodesics in *Sol* geometry, *Beiträge zur Algebra und Geometrie* **48** (2) (2007) 411–421.

[2] A. Bölcskei, B. Szilágyi, Visualization of curves and spheres in *Sol* geometry, *KoG* **10** (2006) 27–32.

[3] E. Molnár, The projective interpretation of the eight 3-dimensional homogeneous geometries, *Beiträge zur Algebra und Geometrie* **38** (2) (1997) 261–288.

[4] E. Molnár, On *Nil* geometry, *Periodica Polytechnica Mechanical Engineering* **47** (1) (2003) 41–49.

[5] E. Molnár, J. Szirmai, Symmetries in the 8 homogeneous 3-geometries, in *Symmetry: Culture and Science*, Proceedings of the Symmetry Festival (2009), Budapest.

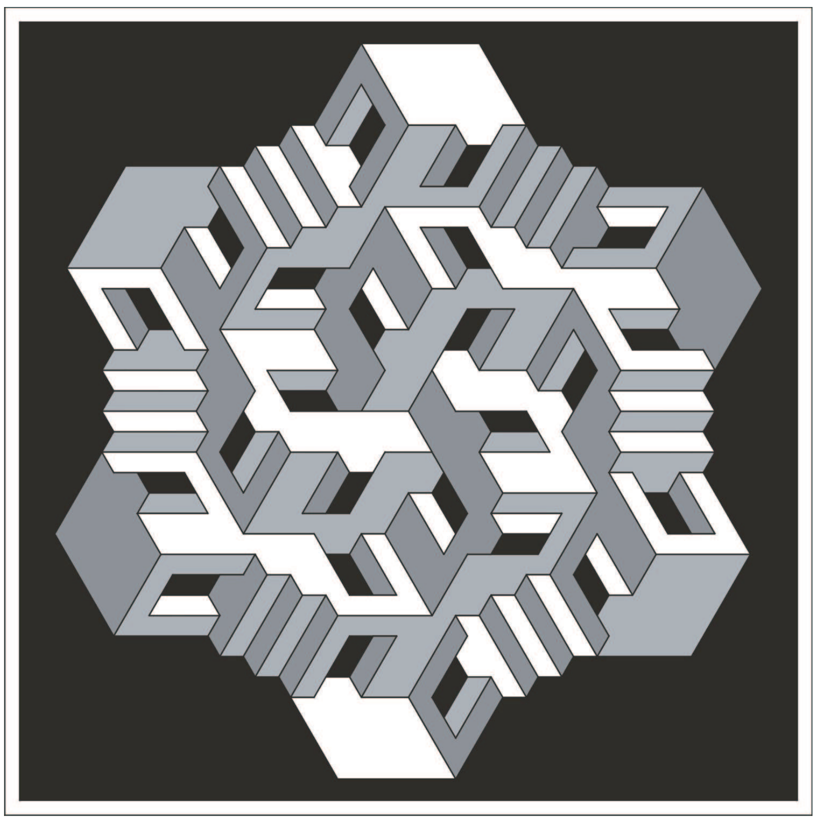
[6] P. Scott, The Geometries of 3-Manifolds, *Bulletin of London Mathematical Society* **15** (1983) 401–487.

From pyramids to hyperspace

Tamás F. Farkas
Miklós Ybl Faculty of Architecture and Civil Engineering, Szent István University, Budapest, Hungary
e-mail: f.farkastamas@freemail.hu

In the presentation I first show my graphics of peculiar spatial objects that seem to be related to pyramids. They can have spiral stairs or simple stairs and they are realizable in 3D. In the second part of the presentation you will see a new series of pictures where higher dimensional objects are modeled on the canvas. You may discover more side-views at the same time and buildings that are like curious spatial mazes.

The aim of the investigation with these pictures is to achieve a better understanding of the three-dimensional space and to develop the spatial ability.



Key words: art and mathematics, spatial education of engineers

MSC 2010: 00A30, 97B40

How a number turns into a zebra – a mathematical photo-shooting

Georg Glaeser
Department for Geometry, University of Applied Arts Vienna, Vienna, Austria
e-mail: georg.glaeser@uni-ak.ac.att

The author will present his latest book with the above title. The contents are mainly about the relationship between biology and mathematics.

The missing link is photography – the book contains about 500 non-trivial photos of animals and plants that shall illustrate the idea behind the concept. To give an example, a typical double page is depicted, showing the largest and smallest land living warm-blooded creatures. Rather simple mathematical considerations (the ratio of volume and surfaces is dependent on the scale, i.e., the absolute size) explain the necessary difference in physiognomy and behavior.

Another sample double page explains how the shallow sea ground can be full of rainbow colors when the sun is getting down. The refraction takes mainly place along certain isophotes of the water surface.

Conics and osculating circles in hyperbolic plane

Helena Halas
Faculty of Civil Engineering, University of Zagreb, Zagreb, Croatia
e-mail: hhalas@master.grad.hr

The Cayley-Klein model is suitable for representing hyperbolic plane for creating geometric constructions, because the projective-geometric points of view in this model for Euclidean and hyperbolic plane are the same. Thus we show the classification of conics in Cayley-Klein model of hyperbolic plane, which can be constructed with perspective collineation as a collinearly related image to the absolute conic. It is shown how to "translate" an Euclidean construction of an osculating circle in an arbitrary point of a conic which is given by general data.

Key words: Cayley-Klein plane, hyperbolic plane, perspective collineation, elation, osculating circle, curvature

MSC 2010: 51N05, 51M15

References

[1] G. Weiss, A. Sliepčević, Osculating circles of conics in Cayley-Klein Planes, *KoG* **13** (2009) 7–12.

On the number of horospheres determined by n points in the hyperbolic space

Antal Joós
College of Dunaújváros, Dunaújváros, Hungary
e-mail: joosanti@gmail.com

The number of the horospheres determined by n points in the 3-dimensional hyperbolic space is at most $2\binom{n}{3}$. We will give a system of n points for every $0 \leq k \leq 2\binom{n}{3}$ such that the number of the horospheres determined by them is equal to k.

Key words: hyperbolic 3-space, horospheres

MSC 2010: 51E30

References

[1] V. Bálint, On a connection between unit circles and horocycles determined by n points, *Periodica Mathematica Hungarica* **38** (1999) 15–17.

[2] V. Bálint, P. Lauron, Improvement of inequalities for the (r, q)-structures and some geometrical connections, *Archivum Mathematicum (Brno)* **31** (1995) 283–289.

[3] J. Beck, On the lattice property of the plane and some problems of Dirac, Motzkin and Erdős in combinatorial geometry, *Combinatorica* **3** (1983) 281–297.

[4] P. Brass, W. Moser, J. Pach, *Research problems in discrete geometry*, Springer Verlag, New York, 2005.

[5] E. Jucovič, Problem 24, in *Combinatorical Structures and their Applications*, New York-London-Paris, Gordon and Breach, 1970.

Clifford's chain of theorems in affine CK-planes

Ema Jurkin
Faculty of Mining, Geology and Petroleum Engineering, University of Zagreb, Zagreb, Croatia
e-mail: ejurkin@rgn.hr

In [2], certain new properties of circles related to the analogues of the Clifford's chain of theorems ([1], p.262) are presented in affine CK-planes. After the first analogue was made, the authors conclude that unfortunately the next step of the Clifford's chain cannot be analogously extended to the affine CK-planes. In this article, by presenting and proving all analogues of the Clifford's chain of theorems in the affine CK-plane, we prove contrary.

Key words: affine Cayley-Klein geometries, Euclidean plane, pseudo-Euclidean plane, isotropic plane, circle, Cliford's chain of theorems

MSC 2010: 51F99, 51M05, 51M99

References

[1] H. S. M. Coxeter, *Introduction to Geometry*, John Wiley and Sons, 2nd ed., 1963.

[2] H. Martini, M. Spirova, Circle geometry in affine Cayley-Klein planes, *Periodica Mathematica Hungarica* **57** (2) (2008) 197 – 206.

[3] H. Sachs, *Ebene Isotrope Geometrie*, Vieweg, Braunschweig-Wiesbaden, 1987.

[4] I. M. Yaglom, *Galilean principle of relativity and Non-Euclidean geometry* (in Russian), Nauka, Moscow, 1969.

Pedal curves of conics in pseudo-Euclidean plane

Mirela Katić-Žlepalo
Department of Civil Engineering, Polytechnic of Zagreb, Zagreb, Croatia
e-mail: mirela.katic-zlepalo@tvz.hr

Ana Sliepčević
Faculty of Civil Engineering, University of Zagreb, Zagreb, Croatia
e-mail: anas@grad.hr

Pedal curves of conics are shown in the projective model of the pseudo-Euclidean plane. Generally, many types of pedal curves in the pseudo-Euclidean plane are analogous to the Euclidean case, but there are some types of pedal curves that we can not get in the Euclidean plane. Those are so called entirely circular quartics and cubics which have different types of singularities in the absolute points. These types of curves are possible in case of:

1) specific position of the pedal point in relation to the conic or the absolute figure or

2) pedal curves of such types of conics that does not exist in the euclidean plane.

Key words: pseudo-Euclidean plane, pedal curves, entirely circular quartic, entirely circular cubic

MSC 2010: 51A051, 51M15

References

[1] N.M. Makarowa, Projektivnie meroopredeljenjija ploskosti, *Uchenye zapiski*, Moskva, (1965) 274–290.

[2] F. Mészaros, Flächen 2-er Ordnung im pseudoeuklidischen Raum $I^{(1)P}$, *Rad HAZU* **[470] 12** (1995) 87–111.

[3] E. Müller, J.L. Krames, *Vorlesungen über darstellende Geometrie*, Bd. II, Leipzig und Wien, 1929.

[4] D. Palman, Vollkommen zirkuläre Kurven 3. Ordnung in der hyperbolischen Ebene, *Glasnik MFA* **14** (1959) 19–74.

[5] N. V. Reveruk, Krivie vtorogo porjadka v psevdoevklidovoi geometrii, *Uchenye zapiski Moskovskogo pedagogicheskogo instituta* **253** (1969) 160–177.

[6] H. Schaal, Euklidische und pseudoeuklidische Sätze über Kreis und gleichseitige Hyperbel, *Elemente der Mathematik* **19** (1964) 53–56.

[7] A. Sliepčević, V. Szirovicza, A Classification and Construction of entirely circular Cubics in the hyperbolic Plane, *Acta Mathematica Hungarica* **104** (3) (2004) 185–201.

[8] V. Szirovicza, Die Fusspunkterzeugung der zirkularen Kurven 3. Ordnung in der isotropen Ebene, *Mathematica Pannonica* **13** (1) (2002) 51–61.

[9] N. Kovačević, Inversion in Minkowskischer Geometrie, *Mathematica Pannonica* **21** (1) (2010) 1–25.

[10] G. Weiss, A. Sliepčević, Osculating Circles of Conics in Cayley-Klein Planes, *KoG* **13** (2009) 7–12.

Thébault circles of the triangle in an isotropic plane

Zdenka Kolar-Begović
Department of Mathematics, Josip Juraj Strossmayer University, Osijek, Croatia
e-mail: zkolar@mathos.hr

Ružica Kolar-Šuper
Faculty of Teacher Education, Josip Juraj Strossmayer University, Osijek, Croatia
e-mail: rkolar@ufos.hr

Vladimir Volenec
Department of Mathematics, University of Zagreb, Zagreb, Croatia
e-mail: volenec@math.hr

In Euclidean geometry V. Thébault considered the existence of the circles which touch the circumscribed circle of a triangle at its vertices as well as the Euler circle of that triangle.

The analogous problem can be studied in the isotropic plane. We are going to prove that there are three circles which touch the circumscribed circle of an allowable triangle at its vertices as well as the Euler circle of that triangle in an isotropic plane. Some statements about relationships between these three circles and some other geometric concepts about triangle have been investigated in an isotropic plane. Formulae for the radii of these circles are also given.

Key words: isotropic plane, allowable triangle, Thébault circles

MSC 2010: 51N25

References

[1] R. Kolar–Šuper, Z. Kolar–Begović, V. Volenec, J. Beban–Brkić, Metrical relationships in a standard triangle in an isotropic plane, *Mathematical Communications* **10** (2005) 149–157.

[2] V. Thébault, Cercles et sphères associés au triangle et au tétraèdre, *Annales de la Société scientifique de Bruxelles*, I, **57** (1937) 54–59; *Mathesis* **51** (1937) 6.

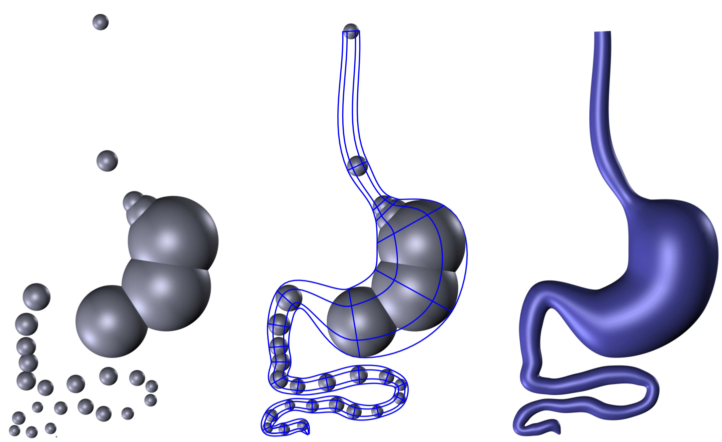
[3] R. Kolar–Šuper, Z.Kolar–Begović, V. Volenec, Thébault circles of the triangle in an isotropic plane, *Mathematical Communications* (accepted for publication).

Skinning of spheres

Roland Kunkli
Department of Computer Graphics and Image Processing, University of Debrecen, Debrecen, Hungary
e-mail: rkunkli@inf.unideb.hu

Miklós Hoffmann
Institute of Mathematics and Informatics, Eszterházy Károly University, Eger, Hungary
e-mail: hofi@ektf.hu

This is a spatial extension of skinning circles given in the talk M. Hoffmann and R. Kunkli "Skinning of circles by Hermite Interpolation and Biarcs". Here an ordered set of spheres is given and the problem is to find a surface which touches all the given spheres. The main point of the algorithm is the computation of the touching circles on the spheres, which are obtained by the spatial extension of the planar algorithm, using Dupin cyclides. Then the Hermite interpolation is applied to define the final surface. Results are compared to existing methods.



Key words: skinning, spheres, Hermite interpolation, Apollonius circles, Dupin cyclides

MSC 2010: 68U05

References

[1] M. Peternell, B. Odehnal, M.L. Sampoli, On quadratic two-parameter families of spheres and their envelopes, *Computer Aided Geometric Design* **25** (2008) 342–355.

[2] G. Slabaugh, B. Whited, T. Fang, J. Rossignac, T. Fang, G. Uzal, 3D ball skinning using PDEs for generation of smooth tubular surfaces, *Computer-Aided Design* **42** (2010) 18–26.

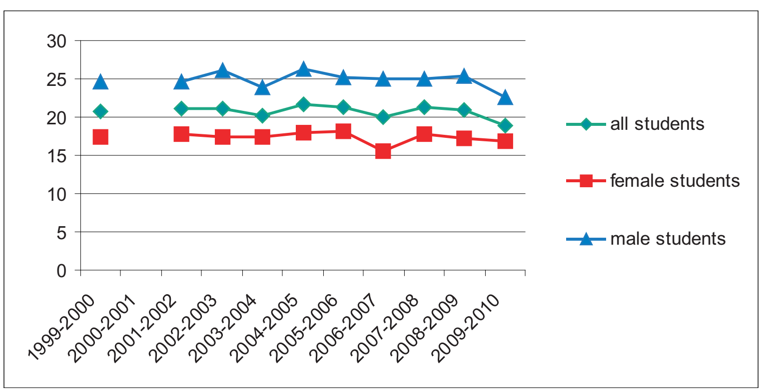
[3] M. Hoffmann, R. Kunkli, Skinning of circles and spheres, *Computer Aided Geometric Design* (submitted)

Ten years of monitoring student's spatial ability at the Faculty of Architecture in Ljubljana

Domen Kušar
Faculty of Architecture, University of Ljubljana, Ljubljana, Slovenia
e-mail: domen.kusar@arh.uni-lj.si

The perception of space is a significant faculty of good architects. At the Faculty of Architecture of the Ljubljana University, we develop spatial perception within the framework of various courses. Since 1999, on the Descriptive Geometry course, we have monitored the level of spatial perception at the outset of the course. For testing, we used the mental rotation test (MRT). Testing conditions have not changed, which ensures the possibility of real comparisons between generations. So far, the results have confirmed the already known and proven differences between the sexes. Some 1,554 students of both sexes participated in the research. The results of the introductory tests from 1999 to 2008 show very minor fluctuations in the level of spatial perception. But the results achieved in the autumn of 2009 exhibited a substantially worse perception of space, which was statistically characteristic of the male students only. We are not aware of the causes of this decline , but some possibilities are suggested in the article. Because of the difficulties of understanding the subject matter, we wanted to know whether the spatial perception was related to success on the course. The investigation disproved the hypothesis. However, the question remains as to whether this lowering of the spatial perception level is a singular example or part of a long-term process. Further investigations will answer this question.

Key words: descriptive geometry, perception of space, mental rotation test (MRT), education
MSC 2010: 97G80, 97G99



Graph: Points achieved at MRT by years (max=40).

Kiepert conics in Cayley-Klein geometries

Sybille Mick
Institute of Geometry, Graz University of Technology, Graz, Austria
e-mail: mick@tugraz.at

In the Euclidean plane the construction of the *Kiepert hyperbola* starts with three isosceles triangles $A'BC$, $AB'C$ and ABC', placed on the sides of ABC with a common base angle α. The lines AA', BB' and CC' share a point $K(\alpha)$. The triangles ABC and $A'B'C'$ are perspective from the centre $K(\alpha)$ and some appropriate axis $d(\alpha)$. If the base angle α varies between $-\pi/2$ and $\pi/2$, the locus of $K(\alpha)$ is the *Kiepert hyperbola* k; the *Kiepert parabola* is defined as the envelope of the axes $d(\alpha)$. The *Kiepert conics* have some connections with other remarkable points and lines of the triangle. Here analogues of *Kiepert conics* in some *Cayley-Klein geometries* are presented and some centres and lines of the triangle related to *Kiepert conics* are investigated.

Key words: Cayley-Klein geometries, triangle, isogonal transformation, Kiepert conics

MSC 2010: 51N05, 51N15, 51F99

References

[1] R. H. Eddy, R. Fritsch, The Conics of Ludwig Kiepert, A Comprehensive Lesson in the Geometry of the Triangle, *Mathematics Magazine* **67** (3) (1997) 188–205.

Translation surfaces in a simply isotropic space

Željka Milin-Šipuš
Department of Mathematics, University of Zagreb, Zagreb, Croatia
e-mail: milin@math.hr

We study translation in the special ambient space – the simply isotropic space. A translation surface is a surface that can locally be written as the sum of two curves. We are specially interested in the analogues of the results from the Euclidean space concerning translation surfaces having constant Gaussian or mean curvature. Furthermore, we are also interested in translation Weingarten surfaces.

Key words: simply isotropic space, translation surface, Weingarten surface

MSC 2010: 53A35

References

[1] H. Liu, Translation surfaces with constant mean curvature in 3-dimensioanl spaces, *Journal of Geometry* **64** (1999) 141–149.

[2] H. Sachs, *Isotrope Geometrie des Raumes*, Vieweg, Braunschweig/Wiesbaden, 1990.

[3] Ž. Milin-Šipuš, B. Divjak, Translation surfaces in the Galilean space, (submitted).

[4] K. Strubecker, Über die isotrope Gegenstücke der Minimalfläche von Scherk, *Journal für die reine und angewandte Mathematik* **293/294** (1977) 22–51.

Cubic grid frameworks

Gyula Nagy
Miklós Ybl Faculty of Architecture and Civil Engineering, Szent István University, Budapest, Hungary
e-mail: nagy.gyula@ybl.szie.hu

In this lecture we add diagonal braces to some of the squares of the square grid framework in such a way as to make the framework rigid. This square grid bracing problem could be solved generally by Maxwell. We show some other results, that are better from an algorithmic point of view. The corresponding problem for cubic grids is open.

Key words: tiling, cubic grid, rigidity, complexity

MSC 2010: 52C25

References

[1] Gy. Nagy, The Rigidity of Special D Cube Grids, *Annales Universitatis Scientarium Budapestinensis* **39** (1996) 107–112.

[2] A. Recski, *Matroid Theory and its Applications in Electric Network Theory and in Statics*, Akadémiai Kiadó, Budapest and Springer, Berlin, 1989.

Equioptic curves (of conic sections)

Boris Odehnal
Institute of Discrete Mathematics and Geometry, Vienna University of Technology, Vienna, Austria
e-mail: boris@geometrie.tuwien.ac.at

The locus e of points where two arbitrary plane curves c_1 and c_2 can be seen under equal angle shall be called *equioptic curve* $e(c_1, c_2)$.

Assuming that both c_1 and c_2 are given by an implicit equation we define $e(c_1, c_2)$ by means of algebraic equations. As an immediate consequence $e(c_1, c_2)$ is algebraic if c_i, $i = 1, 2$ are. We can derive an upper bound on the algebraic degree of the equioptic curve $e(c_1, c_2)$.

The advantages and disadvantages of the algebraic definition of the equioptic curves are discussed. In comparison to the algebraic definition we also give a geometric definition of the equioptic curve. This allows to interpret the equioptic curve $e(c_1, c_2)$ as a projection of the intersection of two level set surfaces. Further, we try to find the number and type of singularities on the algebraic equioptic curves.

In order to give examples we focus our attention to the seemingly simple case of equioptic curves of conic sections.

Key words: equioptic curve, isoptic curve, orthoptic curve, algebraic curve, singularity, conic section

MSC 2010: 51N35

Natural fractals at the middle Adriatic coast

Lidija Pletenac
Faculty of Civil Engineering, University of Rijeka, Rijeka, Croatia
e-mail: pletenac@gradri.hr

A fractal is a geometric shape having fractal dimension that is always greater than its topological dimension. There are three basic types of fractals: geometric, algebraic, and stochastical fractals. There are many organisms in the nature having structure of fractals in their shapes. In this talk the examples of living world are studied, especially plants of middle Adriatic coast that have developed these shapes of fractals, optimum for accomodation to the nature.

Key words: fractals, natural shapes

MSC 2010: 28A80

References

[1] B. B. Mandelbrot, *Fractals and Chaos*, Springer-Verlag, New York, 2004.

[2] B. B. Mandelbrot, *The Fractal Geometry of Nature*, H. B. Fenn and Company Ltd., Canada, 1977.

[3] M. F. Barnsley, *Fractals Everywhere*, Morgan Kaufmann, 2000.

A remarkable overconstrained chain of 16 tetrahedra

Otto Röschel, Peter Fazekas
Institute of Geometry, Graz University of Technology, Graz, Austria
e-mail: roeschel@tugraz.at, fazekas@tugraz.at

We start with a chain of 16 congruent regular tetrahedra which was studied by H. Harborth and M. Möller in [1]. From a kinematic point of view it consists of 16 rigid bodies (the tetrahedra) which are linked via 32 spherical joints. In [1] the authors presented a saturated packing of these tetrahedra without self-intersections - every vertex is linked to some vertex of another tetrahdron. Figure 1 displays the situation - the spherical joints are denoted by small spheres.

The theoretical degree of freedom of this kinematic chain takes on the value $F = -6$, but surprisingly it admits at least a two-parametric self-motion. This self-motion will be studied in this presentation.

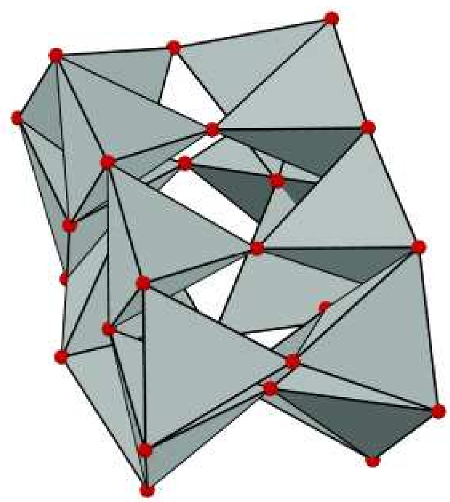


Figure 1: The figure displays the saturated packing of 16 tetrahedra. The bullets symbolize the spherical joints for the self-motion.

Key words: saturated packings of unit tetrahedra, overconstrained mechanisms, self-motions of mechanisms.

MSC 2010: 53A17

References

[1] H. Harborth, M. Möller, Saturated Vertex-to-Vertex Packings of Unit Tetrahedra, *Geombinatorics*, **17** (2) (2007), 53–56.

A kinematic approach to Bäcklund transforms of pseudospherical principal contact element nets

Hans-Peter Schröcker
Unit for Geometry and CAD, University of Innsbruck, Innsbruck, Austria
e-mail: hans-peter.schroecker@uibk.ac.at

A principal contact element net is a map from $\mathbb{Z}^d$ to the space of oriented contact elements (p, N) (point plus oriented normal line) such that any two neighbouring contact elements have a common tangent sphere [1]. The Gaussian curvature of an elementary quadrilateral is defined as the ratio of oriented areas of the Gauss image induced by the normal N and the circular quadrilateral formed by the points p. If it attains a constant negative value throughout the net, we speak of a pseudospherical principal contact element net [2].

Two pseudospherical principal contact element nets are related by a Bäcklund transform if

- corresponding points are at a constant distance,
- corresponding normals form a constant angle, and
- corresponding tangent planes intersect in the connecting line of the corresponding points.

We describe a geometric construction of Bäcklund pairs, based on the geometry of rotation quadrilaterals.

Moreover, we show that Bäcklund pairs satisfy the same permutation property as in the smooth case: Given two Bäcklund pairs $\{A, B\}$ and $\{A, C\}$, there exists a unique pseudospherical principal contact element net D such that $\{B, D\}$ and $\{C, D\}$ are Bäcklund pairs. Our proof is again of kinematic nature and uses the mobility of the Bennett mechanism.

Key words: principal constant element net, Gaussian curvature, pseudospherical surface, rotating motion, Bäcklund transformation

MSC 2010: 53A05, 53A17, 51K10

References

[1] A. I. Bobenko, Y. B. Suris, On organizing principles of discrete differential geometry. Geometry of spheres, *Russian Mathematical Surveys* **62** (1) (2007) 1–43.

[2] A. I. Bobenko, H. Pottmann, J. Wallner, A curvature theory for discrete surfaces based on mesh parallelity, *Mathematische Annalen* **348** (1) (2010) 1–24.

[3] H.-P. SCHRÖCKER, Discrete gliding along principal curves. Accepted for publication in *Proceedings of the 14th International Conference on Geometry and Graphics (ICGG 2010)*, 2010.

[4] H.-P. SCHRÖCKER, Contributions to four-positions theory with relative rotations. In D. L. Pisla, M. Ceccarelli, M. Husty, and Corves. B. J., editors, *New Trends in Mechanism Science. Analysis and Design*, volume 5 of *Mechanism and Machine Science*, pages 21–28. Springer, 2010.

Hermite curves with given curvature and generalization for surfaces

Tibor Schwarcz
Faculty of Informatics, University of Debrecen, Debrecen, Hungary
e-mail: schwarcz@inf.unideb.hu

The starting point of this paper is the degree-5 Hermite Interpolation. It is a widely-known and simple method to extend the basic idea of Hermite interpolation to polynomials of higher degree.

The normal way of defining a curve is giving the endpoints, the two extreme tangent vectors, and the two extreme second derivates. If we denote these geometrical dates by a row vector $\mathbf{G} = [\mathbf{G}_1\ \mathbf{G}_2\ \mathbf{G}_3\ \mathbf{G}_4\ \mathbf{G}_5\ \mathbf{G}_6]$, then the curve is described as $\mathbf{Q}(t) = \mathbf{GMT}$, where $\mathbf{T} = [t^5\ t^4\ t^3\ t^2\ t\ 1]^\top$, $t \in [0,1]$ and the $\mathbf{M}$ is a 6×6 type real matrix, which is easy to calculate.

In this paper we show a simple method to determine the two extreme second derivates, if the osculating circles are given in the points. By this method it is possible to construct an interpolate curve for given points, given tangent vectors and curvatures (osculating circles) in each point. Obviously, in the connection points the segments will have second order continuity and in practice this method can be more convenient for designers than the original one.

In the second part I show how to apply the results to surfaces. The general form of such type of tensor-product surfaces is the following:

$$\mathbf{r}(u,v) = \mathbf{U}^\top \cdot \mathbf{M}^\top \cdot \mathbf{G} \cdot \mathbf{M} \cdot \mathbf{V},$$

where $\mathbf{G}$ is a 6×6 matrix which stores the geometrical data of the surface.

$$\mathbf{G} = \begin{bmatrix} \mathbf{r}(0,0) & \mathbf{r}(0,1) & \dot{\mathbf{r}}_v(0,0) & \dot{\mathbf{r}}_v(0,1) & \ddot{\mathbf{r}}_{v^2}(0,0) & \ddot{\mathbf{r}}_{v^2}(0,1) \\ \mathbf{r}(1,0) & \mathbf{r}(1,1) & \dot{\mathbf{r}}_v(1,0) & \dot{\mathbf{r}}_v(1,1) & \ddot{\mathbf{r}}_{v^2}(1,0) & \ddot{\mathbf{r}}_{v^2}(1,1) \\ \dot{\mathbf{r}}_u(0,0) & \dot{\mathbf{r}}_u(0,1) & \ddot{\mathbf{r}}_{u,v}(0,0) & \ddot{\mathbf{r}}_{u,v}(0,1) & \dddot{\mathbf{r}}_{u,v^2}(0,0) & \dddot{\mathbf{r}}_{u,v^2}(0,1) \\ \dot{\mathbf{r}}_u(1,0) & \dot{\mathbf{r}}_u(1,1) & \ddot{\mathbf{r}}_{u,v}(1,0) & \ddot{\mathbf{r}}_{u,v}(1,1) & \dddot{\mathbf{r}}_{u,v^2}(1,0) & \dddot{\mathbf{r}}_{u,v^2}(1,1) \\ \ddot{\mathbf{r}}_{u^2}(0,0) & \ddot{\mathbf{r}}_{u^2}(0,1) & \dddot{\mathbf{r}}_{u^2,v}(0,0) & \dddot{\mathbf{r}}_{u^2,v}(0,1) & \ddddot{\mathbf{r}}_{u^2,v^2}(0,0) & \ddddot{\mathbf{r}}_{u^2,v^2}(0,1) \\ \ddot{\mathbf{r}}_{u^2}(1,0) & \ddot{\mathbf{r}}_{u^2}(1,1) & \dddot{\mathbf{r}}_{u^2,v}(1,0) & \dddot{\mathbf{r}}_{u^2,v}(1,1) & \ddddot{\mathbf{r}}_{u^2,v^2}(1,0) & \ddddot{\mathbf{r}}_{u^2,v^2}(1,1) \end{bmatrix}$$

I will show some tools to modify the shape of the Hermite tensor-surface. The use of this method makes it possible to connect patches in second order continuity.

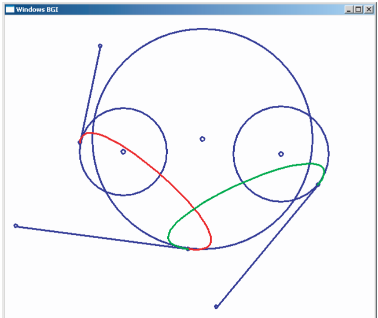


Figure 1

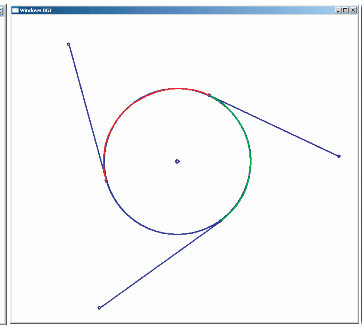


Figure 2

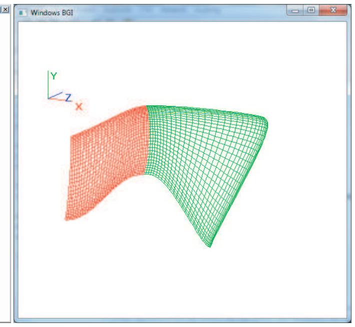


Figure 3

These figures above show some examples. In Figure 1 two connecting segments are drawn. In Figure 2 three connecting segments are drawn with identical osculating circles. In Figure 3 two connecting patches can be seen.

Measurement of the development of spatial ability of Hungarian engineering students -questions and results-

Csilla Sörös
Department of Descriptive Geometry and Computer Science, Szent István University, Budapest, Hungary
e-mail: soros.csilla@ybl.szie.hu

Spatial visualization of engineering students is of greatest importance in terms of their professional achievement. Thus evaluation of this skill is essential. In our University we made a measurement among the students by the help of some well-known (e.g. MCT, MRT) and less favorable spatial ability tests. (The Mental Cutting Test, MCT, is one of the most widely used evaluation method of spatial abilities, see e.g. [2].)

In this lecture I present the results of this research, with special emphasis on long-term developments, and gender differences, extending our previous results [1], [5]. Our questions are the following:
How can we improve the spatial ability after the age of 18? Is it really true that we can observe significant difference between female and male students? What is principally important for the architects and civil engineers?

We compared our results to other international projects, that I discuss in this presentation as well.

References

[1] B. Németh, M. Hoffmann, Gender differences in spatial visualization among engineering students, *Annales Mathematicae et Informaticae* **33** (2006) 169–174.

[2] R. Gorska, Spatial imagination - an overview of the longitudinal research at Cracow University of Technology, *Journal for Geometry and Graphics* **9** (2005) 201–208.

[3] E. Tsutsumi, K. Shiina, A. Suzaki, K. Yamanouchi, S. Takaaki, K. Suzuki, A Mental Cutting Test on female students using a stereographic system, *Journal for Geometry and Graphics* **3** (1999) 111–119.

[4] R.B. Guay, *Purdue Spatial Visualization Test*, Purdue Research Foundation, 1976.

[5] B. Németh, C. Sörös, M. Hoffmann, Typical mistakes in Mental Cutting Test and their consequences in gender differences, *Teaching of Mathematics and Computer Science* **5** (2) (2007) 385–392.

Surface patches constructed from curvature data

Márta Szilvási-Nagy
Department of Geometry, Budapest University of Technology and Economics, Budapest, Hungary
e-mail: szilvasi@math.bme.hu

In this lecture a technique for the construction of smooth surface patches fitted on triangle meshes is presented. Such a surface patch may replace a well defined region of the mesh, and can be used, e.g., in retriangulation and mesh-simplification. Surface patches of several types have been developed for such purposes, mostly quadratic splines or algebraic surfaces. In our algorithm a circular or elliptic surface patch is constructed around a specified triangle face of the mesh, and it is represented by trigonometric vector function. The input data are estimated curvature values and principal directions computed from a ring neighbourhood of the given triangle. The algorithm works on any triangle mesh, also in regions which do not contain triangle vertices. The numerical analysis has been made on synthetic meshes of cylindrical and torodial surfaces. The implementation of the algorithm has been developed in Java programming language and by the help of the symbolical algebraic program package Mathematica.

Key words: surface representations, geometric algorithms

ACM CCS: I.3.5.

References

[1] P.S. Heckbert, M. Garland, Optimal triangulation and quadric-based surface simplification, *Computational Geometry* **14** (1999) 49–65.

[2] L. Kobbelt, M. Botsh, A survay of point-based techniques in computer graphics, *Computers & Graphics* **28** (2004) 801–814.

[3] M. Szilvasi-Nagy, About curvatures on triangle meshes, *KoG* **10** (2006) 13–18.

[4] M. Szilvasi-Nagy, Face-based estimations of curvatures on triangle meshes, *Journal for Geometry and Graphics* **12** (2008) 63–73.

[5] M. Szilvasi-Nagy, Construction of circular splats on analytic surfaces, in *Fifth Hungarian Conference on Computer Graphics and Geometry*, Budapest, 2010, 55–57.

Ball packings in $\mathbf{S}^2 \times \mathbf{R}$ space

Jenö Szirmai
Department of Geometry, Budapest University of Technology and Economics, Budapest, Hungary
e-mail: szirmai@math.bme.hu

W. Thurston classified the eight simply connected 3-dimensional maximal homogeneous Riemannian geometries:

$$\mathbf{E}^3, \mathbf{S}^3, \mathbf{H}^3, \mathbf{S}^2 \times \mathbf{R}, \mathbf{H}^2 \times \mathbf{R}, \widetilde{\mathbf{SL_2R}}, \mathbf{Nil}, \mathbf{Sol}.$$

One of these is $\mathbf{S}^2 \times \mathbf{R}$ geometry which is the direct product of the spherical plane $\mathbf{S}^2$ and the real line $\mathbf{R}$. In [2] J. Z. Farkas has classified and given the complete list the space groups of $\mathbf{S}^2 \times \mathbf{R}$. In this talk we consider the geodesic balls of $\mathbf{S}^2 \times \mathbf{R}$ and compute their volume, define in this space the notion of geodesic ball packing and its density. Moreover, we determine the densest geodesic ball packing for generalized Coxeter space groups of $\mathbf{S}^2 \times \mathbf{R}$. The density of the densest packing for these space groups is 0.82445423. The kissing number of the balls in this packing surprisingly is only 2 (!!).

In our work we use the projective model of $\mathbf{S}^2 \times \mathbf{R}$ introduced by E. Molnár in [1]. The geodesic lines, and geodesic spheres of our space can be visualized on the Euclidean screen of computer. This visualization will show also the arrangement of some geodesic ball packings for the above space groups.

Key words: non-Euclidean geometry, discrete geometry, ball packings

MSC 2010: 52C17, 52C22, 53A20, 51M20

References

[1] E. Molnár: The projective interpretation of the eight 3-dimensional homogeneous geometries, *Beiträge zur Algebra und Geometrie* **38** (2) (1997) 261–288.

[2] J.Z. Farkas: The classification of $\mathbf{S}^2 \times \mathbf{R}$ space groups, *Beiträge zur Algebra und Geometrie* **42** (2001) 235–250.

[3] J. Szirmai: The densest geodesic ball packing by a type of **Nil** lattices, *Beiträge zur Algebra und Geometrie* **48** (2) (2007) 383–398.

[4] J. Szirmai: Geodesic ball packings in $\mathbf{S}^2 \times \mathbf{R}$ space for generalized Coxeter space groups, *Beiträge zur Algebra und Geometrie* (submitted).

Covering a d-dimensional rectangular box by n rectangular boxes of minimum diameter

István Talata
Miklós Ybl Faculty of Architecture and Civil Engineering, Szent István University, Budapest, Hungary
e-mail: talata.istvan@ybl.szie.hu

For a given d-dimensional rectangular box $\prod_{i=1}^{d}[0, a_i]$, let $f_d(B, n)$ be the smallest positive real number for which the rectangular box B can be covered by n rectangular boxes of diameter $f_d(B, n)$. Let $g_d(B, n)$ be the similar quantity with the further restriction that the facets of the boxes in the covering are parallel to the facets of box B. We describe an algorithm that gives an upper bound for $f_d(B, n)$ and $g_d(B, n)$ in terms of d, n and a_i, $1 \leq i \leq d$, recursively, with respect to the dimension d. We compute the exact value of this upper bound in some special cases. We apply the results to get new bounds on a finite sphere packing problem.

Key words: covering, rectangular box, cube, sphere packing

MSC 2010: 52C17, 52C22, 52A40

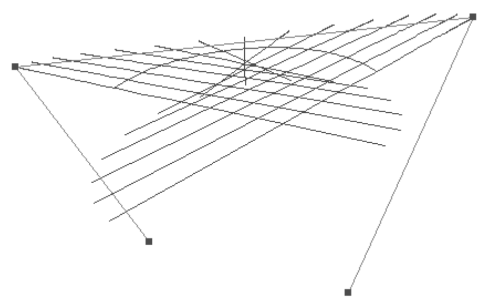
References

[1] V. Bálint, V. Bálint Jr., On the maximum number of points at least one unit away from each other in the unit n-cube, *Periodica Mathematica Hungarica* **57** (1) (2008) 83–91.

Geometric properties and constrained modification of trigonometric spline curves of Han

Ede M. Troll
Eszterházy Károly University, Eger, Hungary
e-mail: ede.troll@gmail.com

New types of quadratic and cubic trigonometric polynomial curves have been introduced in [1] and [2]. These trigonometric curves have a global shape parameter λ. In this paper the geometric effect of this shape parameter on the curves is discussed. We prove that this effect is linear (see Figure). Moreover, we show that the quadratic curve can interpolate the control points at $\lambda = \sqrt{2}$. Constrained modification of these curves is also studied. A curve passing through a given point is computed by an algorithm which includes numerical parts. These results are generalized for surfaces with two shape parameters.



Key words: trigonometric curves, splines, shape parameter, constrained modification

MSC 2010: 68U05

References

[1] X. Han, Quadratic trigonometric polynomial curves with a shape parameter, *Computer Aided Geometric Design* **19** (2002) 503–512.

[2] X. Han, Cubic trigonometric polynomial curves with a shape parameter, *Computer Aided Geometric Design* **21** (2004) 535–548.

Minkowski operations in surface modelling

Daniela Velichová
Mechanical Engineering Faculty, Slovak University of Technology, Bratislava, Slovakia
e-mail: daniela.velichova@stuba.sk

Minkowski operations - sum, product, difference and quotient - of the point sets in the Euclidean space are introduced, and some of their basic properties are presented. Minkowski operations on point sets can be applied as the generating principle in modelling curves, surfaces and solids of interesting forms and properties. There are included several examples of their usage in modelling curve segments and surface patches as results of these operations applied on specific point sets defined by vector representations. Resulting sets of different Minkowski operations on the same input point sets are compared and visualized.

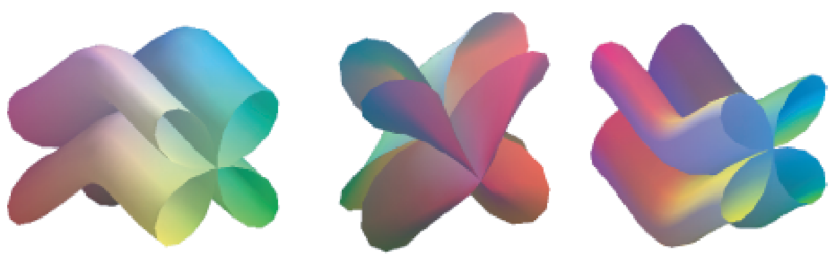


Figure 1: Minkowski sum, product and difference of shamrock curve and versiére.

Key words: Minkowski point set operations, modelling of curves, surfaces and solids

MSC 2010: 51N25, 53A05

References

[1] P. K. A Ghosh, Unified Computational Framework for Minkowski Operations, *Computers & Graphics* **17** (4) (1994) 503–512.

[2] H. K. Hege, K. Polthier, *Visualization and Mathematics III,* Springer Verlag, Berlin, 2003.

[3] M. Smukler, *Geometry, Topology and Applications of the Minkowski Product and Action*, Senior thesis, Harvey Mudd College, Claremont, USA, 2003.

A tool for designing central projections of cubes

László Vörös
M. Pollack Faculty of Engineering, University of Pécs, Pécs, Hungary
e-mail: vorosl@witch.pmmf.hu

A projection plane and a cube in general position are given. The trace points of the significant lines and the trace lines of the planes of the cube compose an acute triangle with special points. In central projection, the vanishing points and lines of the above elements create a configuration similar to the former one. This can be given on the planar intersections of the cube as well, if the planes are parallel with the projection plane. The central projections of these configurations are similar to the former ones. We may apply such a shadow as a construction tool and we can gain the perspective picture of the grid of space-filling mosaics of cubes without taking vanishing points. In this way, we may construct the wanted picture as big as the whole area of the drawing paper.

Our method can be applied in cases corresponding to the practical perspectives with one, two, or three real vanishing points of the edges of the represented cube and we can gain in special cases axonometric projections as well. On the base of the known relationships, we can investigate the shadow of the cube by basic constructive steps if it is a real perspective picture or only a central axonometric projection. You can see some references below about the analytic discussion of this problem in chronology. We have the possibility to design the special perspective picture of the cubes' grid in simplified ways. This can be the constructive and logical frame of graphic art works. By adaptation of the delineated tool, projections of parallelepipeds and 3-dimensional models of more-dimensional cubes can be constructed as well.

Key words: constructive geometry, central projection, practical perspective, special cases

MSC 2010: 51N05, 51N15

References

[1] E. Kruppa, Zur achsonometrischen Methode der darstellenden Geometrie, *Sitzungsberichte, Abteilung II, Österreichische Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse* **119** (1910) 487–506.

[2] J. Szabó, H. Stachel, H. Vogel, Ein Satz über die Zentralaxonometrie, *Sitzungsberichte, Abteilung II, Österreichische Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse* **203** (1994) 3–11.

[3] H. Havlicsek, On the Matrices of Central linear Mappings, *Mathematica Bohemica* **121** (1996) 151–156.

[4] T. Schwarz, Centralaxonometric Mapping in Computer-graphics, *Geometria i Grafika Inzynierska 1* (1996) 37–44.

[5] M. Hoffmann, On the Theorems of Central Axonometry, *Journal for Geometry and Graphics* **2** (1997) 151–155.

[6] A. Dür, An algebraic Equation for Central Projection, *Journal for Geometry and Graphics* **7** (2003) 137–143.

[7] H. Stachel, On Arne Dür's Equation Concerning Central Axonometries, *Journal for Geometry and Graphics* **8** (2004) 215–224.

[8] M. Hoffmann, P. Yiu, Moving Central Axonometric Reference Systems, *Journal for Geometry and Graphics* **9** (2005) 127–134.

A data structure to efficiently map points to their closest point on a polyhedra

Günter Wallner
Department of Geometry, University of Applied Arts Vienna, Vienna, Austria
e-mail: wallner.guenter@uni-ak.ac.at

In many different applications we are confronted with the problem that we have to find the coordinate of the closest point on a polygonal surface for a given point p.

In this talk we discuss a data structure and its implementation with which the afore mentioned task can be performed efficiently. The implementation uses a kd-tree [1], a commonly used space partitioning data structure, which reduces the search complexity to $O(\log N)$ instead of the brute-force $O(N)$ search time, to limit the number of ray-triangle intersections. For each relevant triangle an intersection test, as depicted in Figure 1, is performed. Furthermore, some applications where we successfully used the presented data structure will be shown.

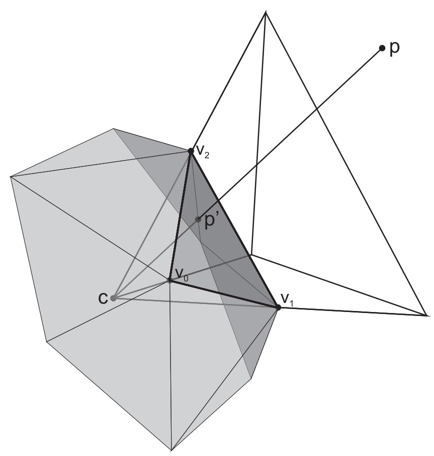


Figure 1: For each triangle $\Delta v_0 v_1 v_2$ a test is performed if the point p is inside the volume bounded by the three planes $\epsilon_i, i = 0...2$ with $\epsilon_i \perp n_i, v_i \in \epsilon_i$ where n_i is the edge normal of $(v_i, v_{(i+1)\%3})$. If this is the case then the intersection point p' of the line $\overline{pc}$ with $\Delta v_0 v_1 v_2$, where c is the intersection point of ϵ_0, ϵ_1 and ϵ_2, is calculated.

Key words: data structures, algorithms, intersection tests, kd-tree, polyhedra

MSC 2010: 68W01

References

[1] J. L. Bentley, Multidimensional binary search trees used for associative searching, *Communications of the ACM* **18** (9) (1975) 509–517.

Uniqueness results for extremal quadrics

Matthias J. Weber
Unit for Geometry and CAD, University of Innsbruck, Innsbruck, Austria
e-mail: Matthias.Weber@uibk.ac.at

It is well known that every full-dimensional compact subset of d-dimensional Euclidean space can be enclosed by a unique ellipsoid of minimal area [1, 2], respectively volume [3, 5]. Recent publications found similar uniqueness results for extremal ellipsoids with respect to other size functions. See [4, 6, 7, 8].

In our talk we present the first uniqueness result in a non-Euclidean geometry. We show that the enclosing conic of minimal area in the elliptic plane is unique if the enclosed set is contained in a circle of diameter $\frac{\pi}{2}$. If the enclosing conics are co-axial or concentric no restrictions have to be made for the enclosed set.

This is a joint work with Hans-Peter Schröcker.

Key words: minimal conic, sphero-conic, spherical ellipse, covering cone, elliptic geometry, spherical geometry.

MSC 2010: 52A40, 52A55, 51M10

This presentation is supported by the Austrian Science Fund (FWF) under grant P21032.

References

[1] F. Behrend, Über einige Affininvarianten konvexer Bereiche, *Mathematische Annalen* **113** (1937) 712–747.

[2] F. Behrend, Über die kleinste umbeschriebene und die größte einbeschriebene Ellipse eines konvexen Bereiches, *Mathematische Annalen* **115** (1938) 397–411.

[3] L. Danzer, D. Laugwitz, H. Lenz, Über das Löwnersche Ellipsoid und sein Analogon unter den einem Eikörper einbeschriebenen Ellipsoiden, *Archiv der Mathematik (Basel)* **8** (3) (1957) 214–219.

[4] P.M. Gruber, Application of an idea of Voronoi to John type problems, *Advances in Mathematics* **218** (2) (2008) 309–351.

[5] F. John, Extremum problems with inequalities as subsidary conditions, in *Studies and Essays. Courant anniversary volume*, Interscience Publ. Inc., New York, 1948, 187–204.

[6] B. Klartag, On John type ellipsoids, in *Geometric Aspects of Functional Analysis, Israel Seminar*, *Springer Lecture Notes in Mathematics* **1850** (2004) 149–158.

[7] H.-P. Schröcker, Uniqueness Results for Minimal Enclosing Ellipsoids, *Computer Aided Geometric Design* **25** (9) (2008) 756–762.

[8] M. J. Weber, H.-P. Schröcker, Davis' Convexity Theorem and Extremal Ellipsoids, *Beiträge zur Algebra und Geometrie* **51** (1) (2010) 263–274.

Line-geometry in Minkowski-spaces

Gunter Weiss
Institute of Geometry, Dresden University of Technology, Dresden, Germany
e-mail: Gunter.Weiss@tu-dresden.de

Classical Minkowski-geometry deals with a normed linear space, whereby the norm is defined by a given (convex) unit ball B and an inner point O as center. Mostly one specializes B to a centrally symmetric, smooth and strictly convex ball and O to its center. As there is, in general, no inner product (in the classical interpretation), in the vector space to such a Minkowski space, there are no rotations, no classical angle measure and many definitions of more or less reasonable orthogonality concepts. One very natural of them, Birkhoff orthogonality, is not symmetric and one distinguishes left- and right-orthogonality. With this orthogonality concept one can try to adapt the Kruppa calculus for ruled surfaces of Euclidean spaces to Minkowski spaces. Here one can define a Minkowski moving frame, a Minkowski striction line s and a Minkowski-covariant differentiation along s as well as Frenet equations for the Minkowski-frame. Another point of view demands the projective extension of the Minkowski space. This allows a treatment based on the usual Klein mapping of the line set to a hyperquadric in a projective 5-space.

Key words: line geometry, Minkowski space, Birkhoff's left-orthogonality, ruled surface, striction curve, pitch, Kruppa's differential invariants

MSC 2010: 53A25, 51M30, 52A15, 51F99

References

[1] H. Pottmann J. Wallner, *Computational Line Geometry*, Springer-Verlag, Berlin Heidelberg New York, 2001.

[2] A.C. Thompson, *Minkowski Geometry*, Cambridge Univ. Press, 1996.

List of participants

1. MAJA ANDRIĆ

   Faculty of Civil Engineering and Architecture,
   University of Split, Split, Croatia

   maja.andric@gradst.hr

2. IVANKA BABIĆ

   Department of Civil Engineering, Polytechnic of Zagreb, Zagreb, Croatia

   ivanka.babic@tvz.hr

3. JELENA BEBAN-BRKIĆ

   Faculty of Geodesy, University of Zagreb, Zagreb, Croatia

   jbeban@geof.hr

4. ATTILA BÖLCSKEI

   Miklós Ybl Faculty of Architecture and Civil Engineering,
   Szent István University, Budapest, Hungary

   bolcskei.attila@ybl.szie.hu

5. IVANA BOŽIĆ

   Department of Civil Engineering, Polytechnic of Zagreb, Zagreb, Croatia

   ivana.bozic@tvz.hr

6. ZDRAVKA BOŽIKOV

   Faculty of Civil Engineering and Architecture,
   University of Split, Split, Croatia

   zdravka.bozikov@gradst.hr

7. ALEKSANDAR ČUČAKOVIĆ

   Faculty of Civil Engineering, University of Belgrade, Belgrade, Serbia

   cucak@grf.bg.ac.rs

8. BLAŽENKA DIVJAK

   Faculty of Organization and Informatics,
   University of Zagreb, Varaždin, Croatia

   blazenka.divjak@foi.hr

9. TOMISLAV DOŠLIĆ

   Faculty of Civil Engineering, University of Zagreb, Zagreb, Croatia

   doslic@grad.hr

10. IVANA ĐURĐEVIĆ

    Faculty of Teacher Education,
    Josip Juraj Strossmayer University, Osijek, Croatia

    idjurdjevic@ufos.hr

11. Zlatko Erjavec

Faculty of Organization and Informatics,
University of Zagreb, Varaždin, Croatia

zlatko.erjavec@foi.hr

12. Tamás F. Farkas

Miklós Ybl Faculty of Architecture and Civil Engineering,
Szent István University, Budapest, Hungary

f.farkastamas@freemail.hu

13. Georg Glaeser

Department of Geometry, University of Applied Arts Vienna, Vienna, Austria

georg.glaeser@uni-ak.ac.at

14. Sonja Gorjanc

Faculty of Civil Engineering, University of Zagreb, Zagreb, Croatia

sgorjanc@grad.hr

15. Ana Gudelj

Faculty of Civil Engineering and Architecture,
University of Split, Split, Croatia

agudelj@gradst.hr

16. Helena Halas

Faculty of Civil Engineering, University of Zagreb, Zagreb, Croatia

hhalas@grad.hr

17. Miklós Hoffmann

Institute of Mathematics and Informatics,
Eszterházy Károly University, Eger, Hungary

hofi@ektf.hu

18. Antal Joós

College of Dunaújváros, Dunaújváros, Hungary

joosanti@gmail.com

19. Biljana Jović

Faculty of Forestry, University of Belgrade, Belgrade, Serbia

ngbilja@afrodita.rcub.bg.ac.rs

20. Ema Jurkin

Faculty of Mining, Geology and Petroleum Engineering,
University of Zagreb, Zagreb, Croatia

ejurkin@rgn.hr

21. Mirela Katić-Žlepalo

Department of Civil Engineering, Polytechnic of Zagreb, Zagreb, Croatia

mirela.katic-zlepalo@tvz.hr

22. Zdenka Kolar-Begović
Department of Mathematics,
Josip Juraj Strossmayer University, Osijek, Croatia
zkolar@mathos.hr

23. Ružica Kolar-Šuper
Faculty of Teacher Education,
Josip Juraj Strossmayer University, Osijek, Croatia
rkolar@ufos.hr

24. Roland Kunkli
Department of Computer Graphics and Image Processing,
University of Debrecen, Debrecen, Hungary
kunkli.roland@inf.unideb.hur

25. Domen Kušar
Faculty of Architecture, University of Ljubljana, Ljubljana, Slovenia
domen.kusar@arh.uni-lj.si

26. Sybille Mick
Institute of Geometry, Graz University of Technology, Graz, Austria
mick@tugraz.at

27. Željka Milin-Šipuš
Department of Mathematics, University of Zagreb, Zagreb, Croatia
milin@math.hr

28. Gyula Nagy
Miklós Ybl Faculty of Architecture and Civil Engineering,
Szent István University, Budapest, Hungary
nagy.gyula@ybl.szie.hu

29. Boris Odehnal
Institute of Discrete Mathematics and Geometry,
Vienna University of Technology, Vienna, Austria
boris@geometrie.tuwien.ac.at

30. Ana Perić
Faculty of Teacher Education,
Josip Juraj Strossmayer University, Osijek, Croatia
aperic@ufos.hr

31. Lidija Pletenac
Faculty of Civil Engineering, University of Rijeka, Rijeka, Croatia
pletenac@gradri.hr

32. Mirna Rodić Lipanović
Faculty of Textile Technology, University of Zagreb, Zagreb, Croatia
mrodic@ttf.hr

33. Otto Röschel

    Institute of Geometry, Graz University of Technology, Graz, Austria

    roeschel@tugraz.at

34. Hans-Peter Schröcker

    Unit for Geometry and CAD, University of Innsbruck, Innsbruck, Austria

    hans-peter.schroecker@uibk.ac.at

35. Tibor Schwarcz

    Faculty of Informatics, University of Debrecen, Debrecen, Hungary

    schwarcz@inf.unideb.hu

36. Ana Sliepčević

    Faculty of Civil Engineering, University of Zagreb, Zagreb, Croatia

    anas@grad.hr

37. Gorana Sović

    Faculty of Civil Engineering and Architecture,
    University of Split, Split, Croatia

    gsovic@gradst.hr

38. Csilla Sörös

    Miklós Ybl Faculty of Architecture and Civil Engineering,
    Szent István University, Budapest, Hungary

    soros.csilla@ybl.szie.hu

39. Nikoleta Sudeta

    Faculty of Architecture, University of Zagreb, Zagreb, Croatia

    nikoleta.sudeta@arhitekt.hr

40. Márta Szilvási-Nagy

    Department of Geometry,
    Budapest University of Technology and Economics, Budapest, Hungary

    szilvasi@math.bme.hu

41. Jenö Szirmai

    Department of Geometry,
    Budapest University of Technology and Economics, Budapest, Hungary

    szirmai@math.bme.hu

42. Vlasta Ščurić-Čudovan

    Faculty of Geodesy, University of Zagreb, Zagreb, Croatia

43. Marija Šimić

    Faculty of Architecture, University of Zagreb, Zagreb, Croatia

    msimic@arhitekt.hr

44. István Talata

Miklós Ybl Faculty of Architecture and Civil Engineering,
Szent István University, Budapest, Hungary

talata.istvan@ybl.szie.hu

45. Ede Troll

Eszterházy Károly University, Eger, Hungary

ede.troll@gmail.com

46. Daniela Velichová

Mechanical Engineering Faculty,
Slovak University of Technology, Bratislava, Slovakia

daniela.velichova@stuba.sk

47. László Vörös

M. Pollack Faculty of Engineering, University of Pécs, Pécs, Hungary

vorosl@witch.pmmf.hu

48. Günter Wallner

Department of Geometry, University of Applied Arts Vienna, Vienna, Austria

wallner.guenter@uni-ak.ac.at

49. Matthias J. Weber

Unit for Geometry and CAD, University of Innsbruck, Innsbruck, Austria

Matthias.Weber@uibk.ac.at

50. Gunter Weiss

Institute of Geometry, Dresden University of Technology, Dresden, Germany

Gunter.Weiss@tu-dresden.de

51. Norman John Wildberger

University of New South Wales, Sydney, Australia

n.wildberger@unsw.edu.au